Laplace Transforms

Piece-Wise Continuous (or) Sectional Continuity:

A function $f(x)$ is called Piece-wise continuous (or) sectionally continuous in any interval $[a, b]$ if $f(x)$ is continuous and has finite left and right hand limits in every subinterval $[a_1, b_1]$ of $[a, b]$ as shown in the following graph.

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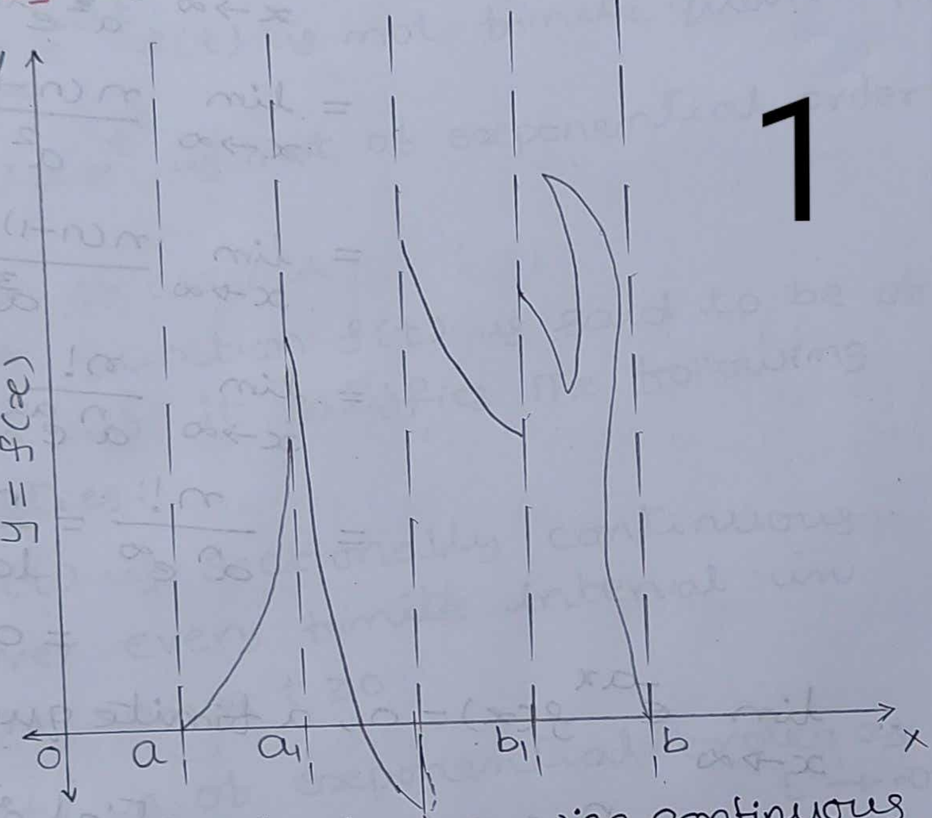

Fig: Graph of piece-wise continuous function

Function of Exponential order:

A function $f(x)$ is said to be of exponential order a as $x \to \infty$ if

$$\lim_{x \to \infty} e^{-ax} f(x) = \text{finite quantity.}$$

i.e for a given +ve integer n_0, $\exists$ a real number $M > 0$ such that

$$|e^{-ax} f(x)| < M, \ \forall x \geq n_0$$

i.e $$|f(x)| < Me^{ax}, \ \forall x \geq n_0$$

Ex: ① Show that x^n is of exponential order as $x \to \infty$ where n being any +ve integer.

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Sol: Let $f(x) = x^n$, $n \in Z^+$

Now $$\lim_{x\to\infty} e^{-ax} f(x) = \lim_{x\to\infty} e^{-ax} x^n$$

$$= \lim_{x\to\infty} \frac{x^n}{e^{ax}}$$

$$= \lim_{x\to\infty} \frac{n x^{n-1}}{a e^{ax}}$$

$$= \lim_{x\to\infty} \frac{n(n-1) x^{n-2}}{a^2 e^{ax}}$$

$$= \lim_{x\to\infty} \frac{n(n-1)(n-2) x^{n-3}}{a^3 e^{ax}}$$

$$= \lim_{x\to\infty} \frac{n!}{a^n e^{ax}}$$

$$= \frac{n!}{a^n e^{\infty}} = \frac{n!}{\text{large value}}$$

$$= 0$$

$\therefore \lim_{x\to\infty} e^{-ax} f(x) = 0$, a finite quantity

$\therefore f(x) = x^n$ is of exponential order a as $x \to \infty$

Note : L' Hospital Rule :

Suppose $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ can not exist and the limit may be an indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, Then we use L' Hospital rule. which is defined as

$$\boxed{\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{f''(x)}{g''(x)} = \ldots\ldots}$$

Ex: ② Show that $f(t) = e^{t^2}$ is not of exponential order as $t \to \infty$.

Sol: Let $f(t) = e^{t^2}$

Now
$$\lim_{t\to\infty} e^{-at} f(t) = \lim_{t\to\infty} e^{-at} e^{t^2}$$
$$= \lim_{t\to\infty} e^{t^2 - at}$$
$$= \lim_{t\to\infty} e^{(t-a)t} \to \infty$$

$\therefore \lim_{t\to\infty} e^{-at} f(t)$ is not finite quantity

$\therefore f(t) = e^{t^2}$ is not of exponential order.

Function of class A:

A function $f(t)$ is said to be of class A if it satisfies the following properties:

i) $f(t)$ is sectionally continuous over every finite interval in the range $t \geq 0$.

ii) $f(t)$ is of exponential order as $t \to \infty$

Eg: 1) $f(t) = t^n, n \in Z^+$ is a function of class A

2) $f(t) = e^{t^2}$ is not a function of class A.

Laplace Transform:

Suppose $F(t)$ is a real valued function defined over the interval $(-\infty, \infty)$ such that $F(t) = 0, \ \forall \ t < 0$.

The Laplace Transform of $F(t)$ is denoted by $L\{F(t)\}$ and defined as

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt$$

(or)

$$L\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\,dt$$

(or)

$$L\{F(t)\} = f(P) = \int_0^\infty e^{-Pt} F(t)\,dt$$

Here L is called Laplace transform operator and s, P are parameters are either real (or) complex numbers.

Notations:

Here we follow two types of notations

1. Functions are denoted by $F(t), G(t), H(t) \ldots\ldots$ and their Laplace transforms are denoted by $f(s), g(s), h(s), \ldots\ldots$ or $f(P), g(P), h(P), \ldots\ldots$
2. If functions are denoted by lower case letters $f(s), g(s), h(s) \ldots\ldots$, then their Laplace transforms are denoted by $\bar{f}(P), \bar{g}(P), \bar{h}(P) \ldots\ldots$ (or) $\bar{f}(s), \bar{g}(s), \bar{h}(s) \ldots\ldots$

The Gamma Function:

If $n>0$, we define the gamma function as

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

→ $\Gamma(n+1) = n\,\Gamma(n)$ $\{= n!$ if n is +ve integer$\}$

→ $\Gamma(n) = (n-1)!$ if $n \in N$

→ $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

→ $\Gamma(1-n)\,\Gamma(n) = \dfrac{\pi}{\sin \pi n}$

→ $\Gamma(1) = 1$

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Laplace Transform of Standard Functions:

1. Prove that $L(1) = \frac{1}{p}$, $p>0$

Sol: By def of Laplace Transform, we have

$$L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt$$

$\boxed{\int e^{ax}dx = \frac{1}{a}e^{ax}}$

put $F(t) = 1$

$$L\{1\} = \int_0^\infty e^{-pt} dt$$

$$= -\frac{1}{p}\left[e^{-pt}\right]_{t=0}^{t=\infty}$$

$$= -\frac{1}{p}\left[e^{-\infty} - e^{0}\right]$$

$\boxed{e^{-\infty} = 0}$

$$= -\frac{1}{p}[0-1]$$

$$= \frac{1}{p}, \quad p>0$$

$$\therefore L\{1\} = \frac{1}{p}, \quad p>0$$

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2. Prove that $L\{t^n\} = \frac{\Gamma(n+1)}{p^{n+1}}$, $n+1>0$

By definition of Laplace Transform, we have

$$L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt$$

put $F(t) = t^n$

$$L\{t^n\} = \int_0^\infty e^{-pt} t^n\,dt$$

put $pt = x \Rightarrow t = \frac{x}{p}$

$p\,dt = dx \Rightarrow dt = \frac{dx}{p}$

$$= \int_0^\infty e^{-x} \left(\frac{x}{p}\right)^n \frac{dx}{p}$$

$$= \int_0^\infty e^{-x} \frac{x^n}{p^{n+1}}\,dx$$

$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx$

$$= \frac{1}{p^{n+1}} \int_0^\infty e^{-x} x^n\,dx$$

$\Gamma(n+1) = \int_0^\infty e^{-x} x^n\,dx$

$$= \frac{1}{p^{n+1}} \Gamma(n+1)$$

$$\therefore\ L\{t^n\} = \frac{\Gamma(n+1)}{p^{n+1}},\ n+1>0$$

If n is a +ve integer

Then $\Gamma(n+1) = n!$

$$\therefore\ L\{t^n\} = \frac{n!}{p^{n+1}}$$

Note: you can use the parameter s in place of p.

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

3. Prove that $L\{e^{at}\} = \frac{1}{p-a}$ if $p > a$

By definition of Laplace transform, we have

$$L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt$$

put $F(t) = e^{at}$

$$L\{e^{at}\} = \int_0^\infty e^{-pt} e^{at}\,dt$$

$$= \int_0^\infty e^{-(p-a)t}\,dt$$

$$= -\left[\frac{e^{-(p-a)t}}{p-a}\right]_{t=0}^{t=\infty}$$

$$= -\frac{1}{(p-a)}\left[e^{-\infty} - e^{0}\right]$$

$$= -\frac{1}{(p-a)}[0-1]$$

$$= \frac{1}{p-a}$$

$$\therefore L\{e^{at}\} = \frac{1}{p-a}$$

Similarly $L\{e^{-at}\} = \frac{1}{p+a}$

Note:

Cauchy - Euler Formula

$$\left.\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}\right\} \Rightarrow \begin{aligned} e^{iax} &= \cos ax + i\sin ax \\ e^{-iax} &= \cos ax - i\sin ax \end{aligned}$$

Hyperbolic Trigonometric functions

$$\cosh x = \frac{e^{x} + e^{-x}}{2}, \quad \sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\cosh at = \frac{e^{at} + e^{-at}}{2}, \quad \sinh at = \frac{e^{at} - e^{-at}}{2}$$

4. Prove that $L\{\cos at\} = \frac{p}{p^2+a^2}$, $L\{\sin at\} = \frac{a}{p^2+a^2}$

Sol: By def of Laplace Transform, we have

$$L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt$$

put $F(t) = e^{-iat}$

$$L\{e^{-iat}\} = \int_0^\infty e^{-pt} e^{-iat}\,dt$$

$$= \int_0^\infty e^{-(p+ia)t}\,dt$$

$$= -\left[\frac{e^{-(p+ia)t}}{(p+ia)}\right]_{t=0}^{t=\infty}$$

$$= -\left[\frac{e^{-\infty} - e^{0}}{p+ia}\right]$$

$$= -\left[\frac{0-1}{p+ia}\right] = \frac{1}{p+ia}$$

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$$\therefore L\{e^{-iat}\} = \frac{1}{p+ia}$$

$$\Rightarrow L\{\cos at - i\sin at\} = \frac{1}{p+ia} \times \frac{p-ia}{p-ia}$$

$$\Rightarrow L\{\cos at\} - i\,L\{\sin at\} = \frac{p-ia}{p^2+a^2}$$

$$= \frac{p}{p^2+a^2} - i\,\frac{a}{p^2+a^2}$$

Equating the real and imaginary parts, we get

$$L\{\cos at\} = \frac{p}{p^2+a^2} \text{ and } L\{\sin at\} = \frac{a}{p^2+a^2}$$

5. Prove that $L\{\sinh at\} = \dfrac{a}{p^2-a^2}$

$$L\{\sinh at\} = L\left\{\frac{e^{at}-e^{-at}}{2}\right\}$$

$$= \frac{1}{2} L\{e^{at} - e^{-at}\}$$

$$= \frac{1}{2} L\{e^{at}\} - \frac{1}{2} L\{e^{-at}\}$$

$$= \frac{1}{2}\left\{\frac{1}{p-a} - \frac{1}{p+a}\right\}$$

$$= \frac{1}{2}\left\{\frac{(p+a)-(p-a)}{p^2-a^2}\right\}$$

$$= \frac{1}{2}\left\{\frac{2a}{p^2-a^2}\right\} = \frac{a}{p^2-a^2}$$

$$\therefore L\{\sinh at\} = \frac{a}{p^2-a^2}$$

5. Prove that $L\{\cosh at\} = \dfrac{p}{p^2-a^2}$

$$L\{\cosh at\} = L\left\{\frac{e^{at}+e^{-at}}{2}\right\}$$

$$= \frac{1}{2} L\{e^{at} + e^{-at}\}$$

$$= \frac{1}{2}\{L(e^{at}) + L(e^{-at})\}$$

$$= \frac{1}{2}\left\{\frac{1}{p-a} + \frac{1}{p+a}\right\}$$

$$= \frac{1}{2}\left\{\frac{p+a+p-a}{p^2-a^2}\right\}$$

$$= \frac{1}{2}\left\{\frac{2p}{p^2-a^2}\right\} = \frac{p}{p^2-a^2}$$

$$\therefore L\{\cosh at\} = \frac{p}{p^2-a^2}$$

Theorem : Linear Property

Suppose $f_1(s)$ and $f_2(s)$ are laplace transforms of $F_1(t)$ and $F_2(t)$ respectively. Then

$$L\{c_1 F_1(t) + c_2 F_2(t)\} = c_1 L\{F_1(t)\} + c_2 L\{F_2(t)\},$$

where c_1, c_2 are arbitrary constants.

Proof: Given That:

$f_1(s)$ is Laplace transform of $F_1(t)$

i.e $L\{F_1(t)\} = f_1(s) = \int_0^\infty e^{-st} F_1(t)\,dt \rightarrow ①$

and

$f_2(s)$ is Laplace transform of $F_2(t)$

i.e $L\{F_2(t)\} = f_2(s) = \int_0^\infty e^{-st} F_2(t)\,dt \rightarrow ②$

Let c_1, c_2 are arbitrary constants.

Now Consider

$$
\begin{aligned}
L\{c_1 F_1(t) + c_2 F_2(t)\} &= L\{H(t)\} \\
&= \int_0^\infty e^{-st} H(t)\,dt \\
&= \int_0^\infty e^{-st} \left[c_1 F_1(t) + c_2 F_2(t)\right] dt \\
&= \int_0^\infty e^{-st} c_1 F_1(t)\,dt + \int_0^\infty e^{-st} c_2 F_2(t)\,dt \\
&= c_1 \int_0^\infty e^{-st} F_1(t)\,dt + c_2 \int_0^\infty e^{-st} F_2(t)\,dt \\
&= c_1 f_1(s) + c_2 f_2(s) \\
&= c_1 L\{F_1(t)\} + c_2 L\{F_2(t)\}
\end{aligned}
$$

$\therefore L\{c_1 F_1(t) + c_2 F_2(t)\} = c_1 L\{F_1(t)\} + c_2 L\{F_2(t)\}$,

where c_1, c_2 are arbitrary constants.

Note: In general

$$L\left\{\sum_{r=1}^{n} c_r F_r(t)\right\} = \sum_{r=1}^{n} c_r L\{F_r(t)\}$$

Theorem : First shifting Theorem (First Translation Theorem)

If $L\{F(t)\} = f(s)$, then $L\{e^{at}F(t)\} = f(s-a)$

Proof : Given That :

$$L\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\,dt \longrightarrow ①$$

Consider

$$L\{e^{at}F(t)\} = L\{H(t)\} \quad , \text{ where } H(t) = e^{at}F(t)$$

$$= \int_0^\infty e^{-st} H(t)\,dt$$

$$= \int_0^\infty e^{-st} e^{at} F(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)} F(t)\,dt$$

$$= \int_0^\infty e^{-ut} F(t)\,dt$$

$$= \cancel{F(u)}$$

$$= f(u) = f(s-a)$$

$$\therefore L\{e^{at}F(t) = f(s-a)$$

Hence, if $L\{F(t)\} = f(s)$, then $L\{e^{at}F(t)\} = f(s-a)$

Note: First shifting theorem can also be stated as: If $L\{F(t)\} = f(s)$, then

$$L\{e^{-at}F(t)\} = f(s+a)$$

Proof : $L\{e^{-at}F(t)\} = L\{H(t)\}$, $H(t) = e^{-at}F(t)$

$$= \int_0^\infty e^{-st} H(t)\,dt$$

$$= \int_0^\infty e^{-st} e^{-at} F(t)\,dt$$

$$= \int_0^\infty e^{-(s+a)t} F(t)\,dt$$

$$= \int_0^\infty e^{-ut} F(t)\,dt \; ; \quad u = s+a$$

$$= f(u) = f(s+a)$$

$$\therefore L\{e^{-at}F(t)\} = f(s+a)$$

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Theorem: Second shifting theorem

If $L\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a) & \text{if } t>a \\ 0 & \text{if } t<a \end{cases}$

then $L\{G(t)\} = e^{-as} f(s)$

(OR)

$$L\{F(t-a)\,H(t-a)\} = e^{-as} f(s)$$

Proof: Given that:

$$L\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\,dt \longrightarrow (1) \text{ and}$$

$$G(t) = \begin{cases} F(t-a) & \text{if } t>a \\ 0 & \text{if } t<a \end{cases}$$

Now

$$L\{G(t)\} = \int_0^\infty e^{-st} G(t)\,dt$$

$$= \int_0^a e^{-st} G(t)\,dt + \int_a^\infty e^{-st} G(t)\,dt$$

$$= \int_0^a e^{-st} \cdot 0 \cdot dt + \int_a^\infty e^{-st} F(t-a)\,dt$$

$$= \int_a^\infty e^{-st} F(t-a)\,dt$$

put $t-a=p$

$\Rightarrow t = p+a$

$dt = dp$

If $t=a \Rightarrow p=0$

If $t=\infty \Rightarrow p=\infty$

$$= \int_{p=0}^\infty e^{-s(p+a)} F(p)\,dp$$

$$= \int_0^\infty e^{-sp} e^{-sa} F(p)\,dp$$

$$= e^{-sa} \int_0^\infty e^{-sp} F(p)\,dp$$

(from eq (1))

$$= e^{-sa} f(s)$$

$$\therefore L\{G(t)\} = e^{-sa} f(s)$$

Hence proved

— x —

Theorem: Change of Scale Property

If $L\{F(t)\} = f(s)$, Then $L\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$

Proof: Given That:

$$L\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\,dt \longrightarrow ①$$

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Now consider

$$L\{F(at)\} = \int_0^\infty e^{-st} F(at)\,dt \longrightarrow ②$$

put $at = u$

$\Rightarrow t = \frac{u}{a}$

$\Rightarrow dt = \frac{1}{a} du$

Limits: $t = 0 \Rightarrow u = 0$

$t = \infty \Rightarrow u = \infty$

Then eq ② $\Rightarrow$

$$L\{F(at)\} = \int_0^\infty e^{-s\left(\frac{u}{a}\right)} F(u) \frac{du}{a}$$

$$= \frac{1}{a} \int_0^\infty e^{-\left(\frac{s}{a}\right)u} F(u)\,du$$

$$= \frac{1}{a} f(s/a)$$

$$\therefore L\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$$

Hence Proved

—×—

Problems based on First Shifting Theorem :

1. Find $L\{t^n e^{at}\}$

Sol : Let $F(t) = t^n$

Then $L\{F(t)\} = L\{t^n\}$

$$= \frac{\Gamma(n+1)}{s^{n+1}} = f(s), \text{ say}$$

By shifting Theorem

"If $L\{F(t)\} = f(s)$, then $L\{e^{at} F(t)\} = f(s-a)$"

Then $L\{t^n e^{at}\} = f(s-a)$

$$= \frac{\Gamma(n+1)}{(s-a)^{n+1}}$$

$$\therefore L\{t^n e^{at}\} = \frac{\Gamma(n+1)}{(s-a)^{n+1}}$$

If n is a +ve integer, then

$$L\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$$

$\boxed{\text{If } n \in Z^+, \text{ then } \Gamma(n+1) = n!}$

2. Find $L\{e^{bt} \cos at\}$

Let $F(t) = \cos at$

Then $L\{F(t)\} = L\{\cos at\}$

$$= \frac{s}{s^2+a^2} = f(s)$$

$\boxed{L\{\cos at\} = \frac{s}{s^2+a^2}}$

By First shifting theorem :

"If $L\{F(t)\} = f(s)$, Then $L\{e^{at} F(t)\} = f(s-a)$"

$$\therefore L\{e^{bt} \cos at\} = f(s-b)$$

$$= \frac{s-b}{(s-b)^2+a^2}$$

$$\therefore L\{e^{bt} \cos at\} = \frac{s-b}{(s-b)^2+a^2}$$

3. Find $L\{e^{bt}\sin at\}$

Let $F(t) = \sin at$

$$\boxed{L\{\sin at\} = \frac{a}{a^2+s^2}}$$

Then $L\{F(t)\} = L\{\sin at\}$

$$= \frac{a}{s^2+a^2} = f(s)$$

By First shifting Theorem

if $L\{F(t)\} = f(s)$, then $L\{e^{at}F(t)\} = f(s-a)$

$$\therefore L\{e^{bt}\sin at\} = f(s-b)$$

$$= \frac{a}{(s-b)^2+a^2}$$

$$\therefore L\{e^{bt}\sin at\} = \frac{a}{(s-b)^2+a^2}$$

4. Find $L\{t^5 e^{3t}\}$

Sol: we know that

$$L\{t^n e^{at}\} = \frac{\Gamma(n+1)}{(s-a)^{n+1}}$$

put $n=5$, $a=3$

$$L\{t^5 e^{3t}\} = \frac{\Gamma(5+1)}{(s-3)^{5+1}}$$

$$= \frac{5!}{(s-3)^6}$$

5. Find $L\{t^6 e^{-2t}\}$

we know that

$$L\{t^n e^{at}\} = \frac{\Gamma(n+1)}{(s-a)^{n+1}}$$

put $n=6$, $a=-2$

$$L\{t^6 e^{-2t}\} = \frac{\Gamma(6+1)}{(s+2)^{6+1}}$$

$$= \frac{6!}{(s+2)^7}$$

6. Find $L\{e^{2t} \sin 4t\}$

We know that

$$L\{e^{bt} \sin at\} = \frac{a}{(s-b)^2 + a^2}$$

put $b = 2,\ a = 4$

$$L\{e^{2t} \sin 4t\} = \frac{a}{(s-b)^2 + a^2} = \frac{4}{(s-2)^2 + 4^2}$$

7. Find $L\{e^{2t} \cos 4t\}$

we know that

$$L\{e^{bt} \cos at\} = \frac{(s-b)}{(s-b)^2 + a^2}$$

put $b = 2,\ a = 4$

$$L\{e^{2t} \cos 4t\} = \frac{(s-2)}{(s-2)^2 + 16}$$

(OR)

Let $F(t) = \cos 4t$

Then $L\{F(t)\} = L\{\cos 4t\}$

$$\boxed{L\{\cos at\} = \frac{s}{s^2 + a^2}}$$

$$= \frac{s}{s^2 + 4^2}$$

$$= \frac{s}{s^2 + 16} = f(s)$$

By First shifting theorem

if $L\{F(t)\} = e^{at}$, $f(s)$, then $L\{e^{at} F(t)\} = f(s-a)$

Then $L\{e^{2t} \cos 4t\} = f(s-2)$

$$= \frac{(s-2)}{(s-2)^2 + 16}$$

$$\therefore L\{e^{2t} \cos 4t\} = \frac{s-2}{(s-2)^2 + 16}$$

8. Find $L\{e^{2t}[\cos 4t + 3\sin 4t]\}$

Sol: we know that

$$L\{e^{bt}\cos at\} = \frac{s-b}{(s-b)^2+a^2}, \quad L\{e^{bt}\sin at\} = \frac{a}{(s-b)^2+a^2}$$

Now

$$L\{e^{2t}[\cos 4t+3\sin 4t]\} = L\{e^{2t}\cos 4t\} + 3L\{e^{2t}\sin 4t\}$$

$$= \frac{s-2}{(s-2)^2+4^2} + 3\cdot\frac{4}{(s-2)^2+4^2}$$

$$= \frac{s-2+12}{(s-2)^2+16} = \frac{s+10}{(s-2)^2+16}$$

$$\therefore L\{e^{2t}[\cos 4t+3\sin 4t]\} = \frac{s+10}{(s-2)^2+16}$$

9. Find $L\{\sinh 2t\}$

we know that $L\{\sinh at\} = \frac{a}{p^2-a^2}$

put $a=2$

$$L\{\sinh 2t\} = \frac{2}{p^2-4}$$

10. Find $L\{\cosh 2t\}$

we know that $L\{\cosh at\} = \frac{p}{p^2-a^2}$

put $a=2$

$$L\{\cosh 2t\} = \frac{p}{p^2-4}$$

11. Find $L\{e^{-t}[3\sinh 2t - 5\cosh 2t]\}$

Sol: Let $F(t) = 3\sinh 2t - 5\cosh 2t$

$$L\{F(t)\} = L\{3\sinh 2t - 5\cosh 2t\}$$

$$= 3L\{\sinh 2t\} - 5L\{\cosh 2t\}$$

$$= 3\cdot\frac{2}{p^2-4} - 5\cdot\frac{p}{p^2-4}$$

$$= \frac{6-5p}{p^2-4} = f(p) \text{ say}$$

By First shifting theorem, we have

If $L\{F(t)\} = f(p)$, Then $L\{e^{-at}F(t)\} = f(p+a)$

$\therefore L\{e^{-t}[3\sinh(2t) - 5\cosh(2t)]\} = f(p+1)$

$$= \frac{6-5(p+1)}{(p+1)^2-4}$$

$$= \frac{1-5p}{p^2+2p-3} = \text{Ans.}$$

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Problems based on Formulas:

1. Find $L\{t^2 e^{-at}\}$

we know that $L\{t^n e^{at}\} = \frac{\Gamma(n+1)}{(s-a)^{n+1}}$

put $n=2$ & replace a by $-a$

$$L\{t^2 e^{-at}\} = \frac{\Gamma(2+1)}{(s+a)^{2+1}}$$

$$= \frac{2!}{(s+a)^3}$$

2. Find Laplace transform of $2e^{5t}$ and $2e^{2t}$

we know that $L\{e^{at}\} = \frac{1}{p-a}$

i) $L\{2e^{5t}\} = 2L\{e^{5t}\} = 2 \cdot \frac{1}{p-5} = \frac{2}{p-5}$

ii) $L\{2e^{2t}\} = \frac{2}{p-2}$

iii) $L\{te^{2t}\} = \frac{\Gamma(2)}{(s-2)^2}$ $\quad \boxed{\because L\{t^n e^{at}\} = \frac{\Gamma(n+1)}{(s-a)^{n+1}}}$

$$= \frac{1}{(s-2)^2}$$

iv) $L\{t^3 - t\} = L\{t^3\} - L\{t\}$ $\quad \boxed{L\{t^n\} = \frac{\Gamma(n+1)}{p^{n+1}}}$

$$= \frac{\Gamma(3+1)}{p^{3+1}} - \frac{\Gamma(1+1)}{p^{1+1}}$$

$$= \frac{3!}{p^4} - \frac{1}{p^2}$$

3. Find Laplace transform of $\sin 5t + \cos 3t$

Sol: $L\{\sin 5t + \cos 3t\} = L\{\sin 5t\} + L\{\cos 3t\}$

$$\boxed{L\{\sin at\} = \frac{a}{p^2+a^2}} \qquad = \frac{5}{p^2+5^2} + \frac{p}{p^2+3^2}$$

$$\boxed{L\{\cos at\} = \frac{p}{p^2+a^2}} \qquad = \frac{5}{p^2+25} + \frac{p}{p^2+9}$$

4. Find $L\{\cos 4t \cos 2t\}$

$$L\{\cos 4t * \cos 2t\} = L\{\tfrac{1}{2}[2\cos 4t \cos 2t]\}$$

$$= \frac{1}{2} L\{\cos 6t + \cos 2t\}$$

$$= \frac{1}{2}\{L[\cos 6t] + L[\cos 2t]\}$$

$$= \frac{1}{2}\left\{\frac{p}{p^2+6^2} + \frac{p}{p^2+2^2}\right\}$$

$$= \frac{p}{2}\left\{\frac{1}{p^2+36} + \frac{1}{p^2+4}\right\}$$

$$\boxed{2\cos A \cos B = \cos(A+B) + \cos(A-B)}$$

5. Find $L\{6\sin 2t - 5\cos 2t\}$

$$L\{6\sin 2t - 5\cos 2t\} = 6L\{\sin 2t\} - 5L\{\cos 2t\}$$

$$= 6 \cdot \frac{2}{p^2+2^2} - 5 \cdot \frac{p}{p^2+2^2}$$

$$= \frac{12-5p}{p^2+4}$$

6. Find $L\{e^{-2t} - e^{3t}\} = L\{e^{-2t}\} - L\{e^{3t}\}$

$$= \frac{1}{p+2} - \frac{1}{p-3}$$

$$= \frac{p-3-p-2}{(p-2)(p-3)} = \frac{-5}{(p-2)(p-3)}$$

7. Fin $L\{2t^3 - 6t + 8\} = 2L\{t^3\} - 6L\{t\} + 8L\{1\}$

$$= 2 \cdot \frac{3!}{p^4} - 6 \cdot \frac{1}{p^2} + \frac{1}{p}$$

$$= \frac{12}{p^4} - \frac{6}{p^2} + \frac{1}{p}$$

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8. Find $L\{\sin t \cos t\}$

$$L\{\sin t\cos t\} = \frac{1}{2}L\{2\sin t\cos t\}$$
$$= \frac{1}{2}L\{\sin 2t\}$$
$$= \frac{1}{2}\cdot\frac{2}{p^2+2^2}$$
$$= \frac{1}{p^2+4}$$

$$\boxed{L\{\sin at\} = \frac{a}{p^2+a^2}}$$

9. Evaluate $L\{(1+t)^2\}$

$$L\{(1+t)^2\} = L\{1+2t+t^2\}$$
$$= L\{1\} + 2L\{t\} + L\{t^2\}$$
$$= \frac{1}{p} + \frac{2}{p^2} + \frac{2}{p^3}$$

10. Find $L\{\sin 3t \cos t\}$

$$\boxed{2\sin A\cos B = \sin(A+B)+\sin(A-B)}$$

$$L\{\sin 3t\cos t\} = \frac{1}{2}\{L[2\sin 3t\cos t]\}$$
$$= \frac{1}{2}L\{\sin 4t + \sin 2t\}$$
$$= \frac{1}{2}\left\{\frac{4}{p^2+4^2} + \frac{2}{p^2+2^2}\right\}$$
$$= \frac{2}{p^2+16} + \frac{1}{p^2+4}$$

11. Find $L\{\sin 3t \sin 2t\}$

$$\boxed{2\sin A\sin B = \cos(A-B)-\cos(A+B)}$$

$$L\{\sin 3t\sin 2t\} = \frac{1}{2}L\{2\sin 3t\sin 2t\}$$
$$= \frac{1}{2}L\{\cos t - \cos 5t\}$$
$$= \frac{1}{2}\{L(\cos t) - L(\cos 5t)\}$$
$$= \frac{1}{2}\left\{\frac{p}{p^2+1} - \frac{p}{p^2+5^2}\right\}$$

$$\boxed{L\{\cos at\} = \frac{p}{p^2+a^2}}$$

$$= \frac{p}{2}\left\{\frac{1}{p^2+1} - \frac{1}{p^2+25}\right\}$$
$$= \frac{p}{2}\left\{\frac{p^2+25-p^2-1}{(p^2+1)(p^2+25)}\right\}$$
$$= \frac{12p}{(p^2+1)(p^2+25)}$$

12. Find the following:

i) $\sin^3 t + e^{2t}$ ii) $t^3 - 3t + 2$ iii) $3t - 5$

iv) $t^3 + 5\cos t$ v) $\sin^3 2t$ vi) $\cos^3 2t$

vii) $\cos^2 t$ viii) $3\cosh 5t - 4\sinh 5t$

ix) $3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t$

x) $t^4 + e^{2t} - \cos t$

xi) $t^3 + t - 5e^{2t} + 2\sin 4t$

Some useful Formulas:

1) $\cos^2 t = \dfrac{1 + \cos 2t}{2}$

2) $\cos 3A = 4\cos^3 A - 3\cos A$

$\Rightarrow \cos^3 A = \dfrac{\cos 3A + 3\cos A}{4}$

3) $\sin 3A = 3\sin A - 4\sin^3 A$

$\Rightarrow \sin^3 A = \dfrac{3\sin A - \sin 3A}{4}$

4) $\int f(x)g(x)dx = f(x)\int g(x)dx - \int\left[f'(x)\int g(x)dx\right]dx$

5) $\int e^{ax}dx = \frac{1}{a}e^{ax}$, $\int e^{-ax}dx = -\frac{1}{a}e^{-ax}$

6) $\int e^{ax}\cos bx\,dx = \dfrac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$

7) $\int e^{ax}\sin bx\,dx = \dfrac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$

8) $\sin 0^\circ = 0$, $\sin\frac{\pi}{2} = 1$, $\sin\pi = 0$, $\sin 2\pi = 0$

$\sin\frac{3\pi}{2} = -1$

$\cos 0^\circ = 1$, $\cos\frac{\pi}{2} = 0$, $\cos\pi = -1$, $\cos 2\pi = 1$

9) $e^{-\infty} = 0$

Problems based on change of scale Property:

1. Apply change of scale property to obtain the Laplace transform of $\sinh(3t)$

Sol: Let $F(t) = \sinh t$

$$\boxed{L\{\sinh at\} = \frac{a}{p^2-a^2}}$$

Then $L\{F(t)\} = L\{\sinh t\}$

$$= \frac{1}{p^2-1} = f(p)$$

By change of scale property:

If $L\{F(t)\} = f(p)$, Then $L\{F(at)\} = \frac{1}{a} f\left(\frac{p}{a}\right)$

Now

$$L\{\sinh 3t\} = \frac{1}{3} f\left(\frac{p}{3}\right)$$

$$= \frac{1}{3} \frac{1}{\left(\frac{p}{3}\right)^2-1}$$

$$= \frac{1}{3} \frac{9}{p^2-9} = \frac{3}{p^2-9}$$

$$\therefore L\{\sinh 3t\} = \frac{3}{p^2-9}$$

2. Apply change of scale property to obtain the Laplace transform of $\cos 5t$.

Sol: Let $F(t) = \cos 5t$

Then $L\{F(t)\} = L\{\cos 5t\} = \frac{p}{p^2+5^2}$ } General Method.

$$\boxed{L\{\cos at\} = \frac{p}{p^2+a^2}}$$

(OR)

Let $F(t) = \cos t$

Then $L\{F(t)\} = L\{\cos t\} = \frac{p}{p^2+1} = f(p)$

By change of scale property:

If $L\{F(t)\} = f(p)$, then $L\{F(at)\} = \frac{1}{a} f\left(\frac{p}{a}\right)$

$$\therefore L\{\cos 5t\} = \frac{1}{5} f\left(\frac{p}{5}\right) = \frac{1}{5}\frac{(p/5)}{(p/5)^2+1} = \frac{1}{5}\frac{(p/5)}{\frac{p^2+25}{25}}$$

$$= \frac{1}{5}\left[\frac{p}{5} \times \frac{25}{p^2+25}\right] = \frac{p}{p^2+25}$$

$$\therefore L\{\cos 5t\} = \frac{p}{p^2+25}$$

3. Apply change of scale property to find $L\{e^{-3t}\}$

Sol: Let $F(t) = e^{t}$

Then $L\{F(t)\} = L\{e^{t}\} = \frac{1}{p-1} = f(p)$ $\quad L\{e^{at}\} = \frac{1}{p-a}$

By the change of scale property

If $L\{F(t)\} = f(p)$, then $L\{F(at)\} = \frac{1}{a} f\left(\frac{p}{a}\right)$

Now
$$L\{e^{-3t}\} = -\frac{1}{3} f\left(\frac{p}{-3}\right)$$
$$= -\frac{1}{3} \frac{1}{-\frac{p}{3}-1} = \frac{1}{3}\left(\frac{3}{p+3}\right)$$
$$= \frac{1}{p+3}$$
$$\therefore L\{e^{-3t}\} = \frac{1}{p+3}$$

4. Given that $L\{F(t)\} = \frac{p^2-p+1}{(2p+1)^2(p-2)}$, then find $L\{F(2t)\}$ by applying change of scale property.

Sol: Given that:
$$L\{F(t)\} = \frac{p^2-p+1}{(2p+1)^2(p-2)} = f(p)$$

By change of scale property:

If $L\{F(t)\} = f(p)$, then $L\{F(at)\} = \frac{1}{a} f\left(\frac{p}{a}\right)$

Now $L\{F(2t)\} = \frac{1}{2} f\left(\frac{p}{2}\right) = \frac{1}{2} \frac{\left[\left(\frac{p}{2}\right)^2 - \frac{p}{2} + 1\right]}{(p+1)^2\left(\frac{p}{2}-2\right)}$

$$= \frac{1}{2} \frac{\frac{p^2-2p+4}{4}}{(p+1)^2 \frac{1}{2}(p-4)}$$
$$= \frac{1}{4} \frac{p^2-2p+4}{(p+1)^2(p-4)}$$

5. Apply change of scale property to find the following:

i) $L\{\sinh(5t)\}$ ii) $L\{\cosh(5t)\}$

iii) $L\{e^{5t}\}$

Problems on Second Shifting Theorem:

1. Evaluate $L\{F(t)\}$ where $F(t) = \begin{cases} (t-1)^2 & \text{if } t>1 \\ 0 & \text{if } t<1 \end{cases}$

Sol: Given that:

$$F(t) = \begin{cases} (t-1)^2 & \text{if } t>1 \\ 0 & \text{if } t<1 \end{cases}$$

Let $a=1$ and $G(t) = t^2$

Then
$$F(t) = \begin{cases} G(t-1) & \text{if } t>1 \\ 0 & \text{if } t<1 \end{cases}$$

Now $L\{G(t)\} = L\{t^2\}$

$$= \frac{\Gamma(2+1)}{p^{2+1}}$$

$$= \frac{2!}{p^3} = f(p)$$

By second shifting theorem:

If $L\{F(t)\} = f(p)$, and $G(t) = \begin{cases} F(t-a) & \text{if } t>a \\ 0 & \text{if } t<a \end{cases}$

Then $L\{\cancel{F}(t)\} = e^{-ap} f(p)$

$$= e^{-p} \frac{2!}{p^3}$$

$$= \frac{2e^{-p}}{p^3}$$

2. Evaluate $L\{F(t)\}$, where $F(t) = \begin{cases} \sin\left(t-\frac{2\pi}{3}\right) & \text{if } t>\frac{2\pi}{3} \\ 0 & \text{if } t<\frac{2\pi}{3} \end{cases}$

Given that:
$$F(t) = \begin{cases} \sin\left(t-\frac{2\pi}{3}\right) & \text{if } t>\frac{2\pi}{3} \\ 0 & \text{if } t<\frac{2\pi}{3} \end{cases}$$

Take $a = \frac{2\pi}{3}$ and $G(t) = \sin t$

Then $L\{G(t)\} = L\{\sin t\}$

$$= \frac{1}{p^2+1} = f(p)$$

Now $F(t) = \begin{cases} G(t-\frac{2\pi}{3}) & \text{if } t > \frac{2\pi}{3} \\ 0 \text{ if } t < \frac{2\pi}{3} \end{cases}$

By second shifting theorem

$L\{F(t)\} = e^{-ap} f(P)$, $a = \frac{2\pi}{3}$, $f(P) = \frac{1}{P^2+1}$

$= e^{-\frac{2\pi P}{3}} \cdot \frac{1}{P^2+1}$

3. Find $L\{F(t)\}$, where $F(t) = \begin{cases} \cos(t-\frac{2\pi}{3}) & \text{if } t > \frac{2\pi}{3} \\ 0 \text{ if } t < \frac{2\pi}{3} \end{cases}$

Sol: Given that:

$F(t) = \begin{cases} \cos(t-\frac{2\pi}{3}) & \text{if } t > \frac{2\pi}{3} \\ 0 & \text{if } t < \frac{2\pi}{3} \end{cases}$

Take $a = \frac{2\pi}{3}$, $G(t) = \cos t$

Then $L\{G(t)\} = L\{\cos t\}$

$= \frac{P}{P^2+1} = f(P)$

Now $F(t) = \begin{cases} G(t-\frac{2\pi}{3}) & \text{if } t > \frac{2\pi}{3} \\ 0 \text{ if } t < \frac{2\pi}{3} \end{cases}$

By second shifting theorem

$L\{F(t)\} = e^{-ap} f(P)$; where $a = \frac{2\pi}{3}$

$f(P) = \frac{P}{P^2+1}$

$= e^{-\frac{2\pi P}{3}} \cdot \frac{P}{P^2+1}$

$= \frac{P e^{-\frac{2P\pi}{3}}}{P^2+1}$

4. Find $L\{F(t)\}$, where $F(t) = \begin{cases} \cos(t-\frac{\pi}{3}) & \text{if } t > \frac{\pi}{3} \\ 0 \text{ if } t < \frac{\pi}{3} \end{cases}$

5. Find $L\{F(t)\}$, where $F(t) = \begin{cases} \sin(t-\frac{\pi}{3}) & \text{if } t > \frac{\pi}{3} \\ 0 \text{ if } t < \frac{\pi}{3} \end{cases}$

Problems based on definition of L.T :

1. Find $L\{F(t)\}$, where $F(t) = \begin{cases} 0 & \text{if } 0<t<1 \\ t & \text{if } 1<t<2 \\ 0 & \text{if } t>2 \end{cases}$

<u>Sol</u> :

Given That: $F(t) = \begin{cases} 0 & \text{if } 0<t<1 \\ t & \text{if } 1<t<2 \\ 0 & \text{if } t>2. \end{cases}$

By def, L.T, we have

$$L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt$$

$$= \int_0^1 e^{-pt} F(t)\,dt + \int_1^2 e^{-pt} F(t)\,dt + \int_2^\infty e^{-pt} F(t)\,dt$$

$$= \int_0^1 e^{-pt}\cdot 0 \cdot dt + \int_1^2 e^{-pt}\cdot t \cdot dt + \int_2^\infty e^{-pt}\cdot 0 \cdot dt$$

$$= \int_1^2 t\, e^{-pt}\,dt$$

$$\boxed{\int f(x)\,g(x)\,dx = f(x)\int g(x)\,dx - \int\left[f'(x)\int g(x)\,dx\right]dx}$$

$$\cancel{= t \int g(t)\,dt}$$

$$= t\int_1^2 e^{-pt}\,dt - \int_1^2 \left[1\cdot \int e^{-pt}\,dt\right]dt$$

$$= \left[\frac{t\, e^{-pt}}{-p}\right]_{t=1}^{2} + \int \frac{e^{-pt}}{p}\,dt$$

$$= -\frac{1}{p}\left[2e^{-2p} - e^{-p}\right] - \left[\frac{e^{-pt}}{p^2}\right]_{t=1}^{2}$$

$$= -\frac{1}{p}\left[2e^{-2p} - e^{-p}\right] - \frac{1}{p^2}\left[e^{-2p} - e^{-p}\right]$$

$$= \frac{-2pe^{-2p} + pe^{-p} - e^{-2p} + e^{-p}}{p^2}$$

$$\therefore L\{F(t)\} = \frac{1}{p^2}\left\{(p+1)e^{-p} - (2p+1)e^{-2p}\right\}$$

= Ans.

2. Find Laplace Transform of the function $F(t)$ defined by $F(t) = \begin{cases} 4 & \text{if } 0 < t < 1 \\ 3 & \text{if } t > 1 \end{cases}$

<u>Sol</u>: Given That:

$$F(t) = \begin{cases} 4 & \text{if } 0<t<1 \\ 3 & \text{if } t>1 \end{cases}$$

By def of Laplace Transform, we have

$$L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt$$

$$= \int_0^1 e^{-pt} F(t)\,dt + \int_1^\infty e^{-pt} F(t)\,dt$$

$$= \int_0^1 e^{-pt}\, 4\,dt + \int_1^\infty e^{-pt}\, 3\,dt$$

$$= 4\int_0^1 e^{-pt}\,dt + 3\int_1^\infty e^{-pt}\,dt$$

$$\boxed{\int e^{at}\,dt = \frac{1}{a}e^{at}}$$

$$= 4\left[\frac{e^{-pt}}{-p}\right]_0^1 + 3\left[\frac{e^{-pt}}{-p}\right]_1^\infty$$

$$= \frac{4}{-p}\left[e^{-p} - e^{0}\right] + \frac{3}{-p}\left[e^{-\infty} - e^{-p}\right]$$

$$\boxed{\begin{array}{c} e^{-\infty} = 0 \\ \hline e^{0} = 1 \end{array}}$$

$$= -\frac{4}{p}\left[e^{-p} - 1\right] - \frac{3}{p}\left[0 - e^{-p}\right]$$

$$= \frac{4}{p}(1 - e^{-p}) + \frac{3}{p}e^{-p}$$

$$= \frac{1}{p}\left[4 - 4e^{-p} + 3e^{-p}\right]$$

$$= \frac{1}{4}\left[4 - e^{-p}\right]$$

$$\therefore\ L\{F(t)\} = \frac{1}{4}\left[4 - e^{-p}\right]$$

Imp 3. Find L.T of F(t), where $F(t) = \begin{cases} 2t & \text{if } 0 \le t \le 5 \\ 1 & \text{if } t > 5 \end{cases}$

Sol:

Given that: $F(t) = \begin{cases} 2t & \text{if } 0 \le t \le 5 \\ 1 & \text{if } t > 5 \end{cases}$

We know that by def of L.T;

$$L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt$$

$$= \int_0^5 e^{-pt} F(t)\,dt + \int_5^\infty e^{-pt} F(t)\,dt$$

$$= \int_0^5 e^{-pt}\, 2t\,dt + \int_5^\infty e^{-pt}\, 1\,dt$$

$$= 2\int_0^5 t e^{-pt}\,dt + \left[\frac{e^{-pt}}{-p}\right]_{t=5}^{\infty}$$

$$= 2\left\{ t\,\frac{e^{-pt}}{-p} + \int_0^5 1\,\frac{e^{-pt}}{p}\,dt \right\} - \frac{1}{p}\left[e^{-\infty} - e^{-5p}\right]$$

$$= 2\left\{ \left[\frac{t e^{-pt}}{-p}\right]_0^5 - \left[\frac{e^{-pt}}{p^2}\right]_0^5 \right\} - \frac{1}{p}\left[0 - e^{-5p}\right]$$

$$= 2\left\{ \left[\frac{5e^{-5p} - 0}{-p}\right] - \frac{1}{p^2}\left[e^{-5p} - e^{0}\right] \right\} + \frac{1}{p}e^{-5p}$$

$$= -\frac{10}{p}e^{-5p} - \frac{2}{p^2}e^{-5p} + \frac{2}{p^2} + \frac{1}{p}e^{-5p}$$

$$= -\frac{9}{p}e^{-5p} - \frac{2}{p^2}e^{-5p} + \frac{2}{p^2}$$

$$\therefore L\{F(t)\} = \frac{e^{-5p}}{p}\left[-9 - \frac{2}{p}\right] + \frac{2}{p^2}$$

= Ans.

Note:

$$\int f(x)g(x)\,dx = f(x)\int g(x)\,dx - \int\left[f'(x)\int g(x)\,dx\right]dx$$

Imp 4. Find L.T of $F(t)$, where $F(t) = \begin{cases} \sin t & \text{if } 0 < t < \pi \\ 0 & \text{if } t > \pi \end{cases}$

Sol: Given that:

$$F(t) = \begin{cases} \sin t & \text{if } 0 < t < \pi \\ 0 & \text{if } t > \pi \end{cases}$$

By def of Laplace Transform, we have

$$L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt$$

$$= \int_0^\pi e^{-pt} F(t)\,dt + \int_\pi^\infty e^{-pt} F(t)\,dt$$

$$= \int_0^\pi e^{-pt} \sin t\,dt + \int_\pi^\infty e^{-pt}\, 0\,dt$$

$$= \int_0^\pi e^{-pt} \sin t\,dt$$

$$\boxed{\int e^{-at} \sin bt\,dt = \frac{e^{-at}}{a^2+b^2}\left[-a\sin bt - b\cos bt\right]}$$

$$= \left\{\frac{e^{-pt}}{p^2+1}\left[-p\sin t - \cos t\right]\right\}_0^\pi$$

$$= \left\{\frac{e^{-p\pi}}{p^2+1}\left[-p\sin \pi - \cos \pi\right] - \frac{e^{0}}{p^2+1}\left[-p\sin 0 - \cos 0\right]\right\}$$

$$= \frac{e^{-p\pi}}{p^2+1}\left[0+1\right] - \frac{1}{p^2+1}\left[0-1\right]$$

$$= \frac{e^{-p\pi}}{p^2+1} + \frac{1}{p^2+1} = \frac{1+e^{-p\pi}}{p^2+1}$$

$$\therefore L\{F(t)\} = \frac{1+e^{-p\pi}}{p^2+1}$$

Note: $\sin \pi = 0, \quad \cos \pi = -1$

$\sin 0^\circ = 0, \quad \cos 0^\circ = 1$

5. Find $L\{F(t)\}$, where $F(t) = \begin{cases} (t-1)^2 & \text{if } t>1 \\ 0 & \text{if } 0<t<1 \end{cases}$

Sol:

Given that: $F(t) = \begin{cases} (t-1)^2 & \text{if } t>1 \\ 0 & \text{if } 0<t<1 \end{cases}$

By ~~def~~ def of L.T, we have

$$L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt$$

$$= \int_0^1 e^{-pt} F(t)\,dt + \int_1^\infty e^{-pt} F(t)\,dt$$

$$= \int_0^1 e^{-pt}\, 0\, dt + \int_1^\infty e^{-pt} (t-1)^2 dt$$

$$= \int_1^\infty (t-1)^2 e^{-pt}\,dt$$

$$= (t-1)^2 \int_1^\infty e^{-pt}\,dt - \int_1^\infty 2(t-1)\int \frac{e^{-pt}}{-p}\,dt$$

$$= (t-1)^2 \left[\frac{e^{-pt}}{-p}\right]_1^\infty + \frac{2}{p}\int_1^\infty (t-1)\, e^{-pt}\,dt$$

$$\cancel{= \frac{(t-1)^2}{-p}}$$

$$= 0 + \frac{2}{p}\left\{(t-1)\frac{e^{-pt}}{-p} + \frac{1}{p}\int e^{-pt}\,dt\right\}_1^\infty$$

$$= 0 + \frac{2}{p}\left[\{0\} + \left\{\frac{1}{p}\,\frac{e^{-pt}}{-p}\right\}\right]_1^\infty$$

$$= \frac{2}{p}\left[-\frac{1}{p^2}\left[e^{-\infty} - e^{-p}\right]\right]$$

$$= -\frac{2}{p^3}\left[0 - e^{-p}\right]$$

$$= \frac{2}{p^3} e^{-p}$$

$$\therefore\ L\{F(t)\} = \frac{2}{p^3} e^{-p}$$

6. Find $L\{F(t)\}$, where $F(t) = \begin{cases} e^t & \text{if } 0 \le t \le 1 \\ 0 & \text{if } t > 1 \end{cases}$

Sol:

Given that: $F(t) = \begin{cases} e^t & \text{if } 0 \le t \le 1 \\ 0 & \text{if } t > 1 \end{cases}$

By def of Laplace Transform, we have

$$L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\, dt$$

$$= \int_0^1 e^{-pt} F(t)\, dt + \int_1^\infty e^{-pt} F(t)\, dt$$

$$= \int_0^1 e^{-pt} e^t\, dt + \int_1^\infty e^{-pt}\, 0\, dt$$

$$= \int_0^1 e^{(-p+1)t}\, dt = \frac{1}{-p+1}\left[e^{(-p+1)t}\right]_0^1$$

$$= \cancel{\frac{1}{-p+1}\left(e^{(-p+1)t}\right)_{t=0}}$$

$$= \frac{1}{1-p}\left[e^{-p+1} - e^0\right]$$

$$= \frac{1}{1-p}\left[e^{1-p} - 1\right]$$

$$L\{F(t)\} = \frac{1}{1-p}\left[e^{1-p} - 1\right]$$

7. Find $L\{F(t)\}$, where $F(t) = \begin{cases} 0 & \text{if } 0 < t < 2 \\ 4 & \text{if } t > 2 \end{cases}$

8. Find $L\{F(t)\}$, where $F(t) = \begin{cases} e^t & \text{if } 0 < t < 5 \\ 3 & \text{if } t > 5 \end{cases}$

9. Find $L\{F(t)\}$ where $F(t) = \begin{cases} 1 & \text{if } 0 < t < 2 \\ t & \text{if } t > 2 \end{cases}$

10. Find $L\{F(t)\}$ where $F(t) = \begin{cases} 0 & \text{if } 0 < t < 3 \\ 4 & \text{if } t > 3 \end{cases}$

Theorem: Laplace Transform of Derivatives:

If $L\{F(t)\} = f(s)$, then $L\{F^{n}(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots - s F^{n-2}(0) - F^{n-1}(0)$, where $F^{n}(t) = \frac{d^n}{dt^n}[F(t)]$

Proof: Given that:

$$L\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\,dt$$

Now consider

$$L\{F'(t)\} = \int_0^\infty e^{-st} F'(t)\,dt = \int_0^\infty e^{-st}\left[\frac{d}{dt}F(t)\right]dt$$

$$= \left[e^{-st}F(t)\right]_0^\infty + \int_0^\infty s\, e^{-st} F(t)\,dt$$

$$= \left[e^{-\infty}F(\infty) - e^{0}F(0)\right] + s\int_0^\infty e^{-st}F(t)\,dt$$

$$= -F(0) + s f(s)$$

Imp

$$\therefore \boxed{L\{F'(t)\} = s f(s) - F(0)}$$

$\therefore$ statement is true for $n = 1$

In view of this,

$$L\{G'(t)\} = s\,g(s) - G(0)$$

Take $\quad G(t) = F'(t) \Rightarrow G'(t) = F''(t)$

Now

$$L\{G'(t)\} = L\{F''(t)\}$$

$$\Rightarrow s\,g(s) - G(0) = L\{F''(t)\}$$

$$\Rightarrow s\,L\{G(t)\} - F'(0) = L\{F''(t)\}$$

$$\Rightarrow s\,L\{F'(t)\} - F'(0) = L\{F''(t)\}$$

$$\Rightarrow s\{s f(s) - F(0)\} - F'(0) = L\{F''(t)\}$$

Imp

$$\therefore \boxed{L\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)}$$

$\therefore$ statement is true for $n = 2$

In general

$$L\{F^{n}(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots - sF^{n-2}(0) - F^{n-1}(0)$$

Hence Proved

P. By using the definition of Laplace transform of derivative of $F(t)$, prove that

i) $L\{t\} = \frac{1}{p^2}$ ii) $L\{e^{at}\} = \frac{1}{p-a}$ iii) $L\{-a \sin at\} = \frac{-a^2}{p^2+a^2}$

Sol:

i) Let $F(t) = t \Rightarrow F(0) = 0$

Now $F'(t) = \frac{d}{dt} F(t) = \frac{d}{dt} t = 1$

By def of L.T of derivative of $F(t)$, we have

$$L\{F'(t)\} = p f(p) - F(0)$$

$$\Rightarrow L\{1\} = p L\{F(t)\} - 0$$

$$\Rightarrow \frac{1}{p} = p L\{F(t)\}$$

$$\Rightarrow \frac{1}{p^2} = L\{F(t)\}, \quad F(t) = t$$

$$\therefore L\{t\} = \frac{1}{p^2}$$

ii) Let $F(t) = e^{at} \Rightarrow F(0) = e^0 = 1$

Then $F'(t) = \frac{d}{dt} F(t)$

$$= \frac{d}{dt} e^{at} = a e^{at}$$

By def of L.T of derivatives of $F(t)$, we have

$$L\{F'(t)\} = p f(p) - F(0)$$

$$\Rightarrow L\{a e^{at}\} = p L\{F(t)\} - 1$$

$$\Rightarrow a L\{e^{at}\} = p L\{F(t)\} - 1$$

$$\Rightarrow a \frac{1}{p-a} + 1 = p L\{F(t)\}$$

$$\Rightarrow \frac{a}{p-a} + 1 = p L\{F(t)\}$$

$$\Rightarrow \frac{a + p - a}{p-a} = p L\{F(t)\}$$

$$\Rightarrow \frac{p}{p-a} \times \frac{1}{p} = L\{F(t)\}, \quad F(t) = e^{at}$$

$$\Rightarrow \frac{1}{p-a} = L\{F(t)\}$$

$$\therefore L\{e^{at}\} = \frac{1}{p-a}$$

=Ans.

iii) Let $F(t) = -a \sin at \Rightarrow F(0) = 0$

Then $F'(t) = \frac{d}{dt} F(t) = \frac{d}{dt}[-a \sin at]$

$= -a^2 \cos at \Rightarrow F'(0) = -a^2 \cos 0^{\circ} = -a^2$

$F''(t) = \frac{d}{dt} F'(t)$

$= \frac{d}{dt}[-a^2 \cos at]$

$= -a^2 \frac{d}{dt} \cos at = a^3 \sin at$

By def of L.T of derivative of $F(t)$, we have

$L\{F''(t)\} = p^2 f(p) - pF(0) - F'(0)$

$\Rightarrow L\{a^3 \sin at\} = p^2 L\{F(t)\} - p \cdot 0 + a^2$

$\Rightarrow -a^2 L\{-a \sin at\} = p^2 L\{F(t)\} + a^2$

$\Rightarrow -a^2 L\{F(t)\} = p^2 L\{F(t)\} + a^2$

$\Rightarrow p^2 L\{F(t)\} + a^2 L\{F(t)\} + a^2 = 0$

$\Rightarrow (p^2 + a^2) L\{F(t)\} = -a^2$

$\Rightarrow L\{F(t)\} = \frac{-a^2}{p^2+a^2}$, $F(t) = -a \sin at$

$\therefore L\{-a \sin at\} = \frac{-a^2}{p^2+a^2}$

Theorem: Laplace Transform of Integral:

If $L\{F(t)\} = f(s)$, then $\frac{1}{s} f(s) = L\left\{\int_0^t F(u)\,du\right\}$

Proof:

Given that $L\{F(t)\} = f(s)$

Take $G(t) = \int_0^t F(u)\,du \longrightarrow (1)$

Then $G(0) = \int_0^0 F(u)\,du = 0$

From eq (1): $G'(t) = \frac{d}{dt}[G(t)]$

$$= \frac{d}{dt}\left\{\int_0^t F(u)\,du\right\}$$

$$= F(t)$$

$\therefore G'(t) = F(t)$

By previous theorem, we have

$L\{G'(t)\} = s\,g(s) - G(0)$

$\Rightarrow L\{G'(t)\} = s\,L\{G(t)\}$, where $G(0) = 0$, $L\{G(t)\} = g(s)$

$\Rightarrow L\{F(t)\} = s\,L\{G(t)\}$

$\Rightarrow \frac{1}{s} f(s) = L\left\{\int_0^t F(u)\,du\right\}$

$\therefore L\left\{\int_0^t F(u)\,du\right\} = \frac{1}{s} f(s)$

Hence proved

Note: If $L\{F(t)\} = f(P)$, then $L\left\{\int_0^t F(u)\,du\right\} = \frac{1}{P} f(P)$

Theorem: Multiplication by powers of t

If $L\{F(t)\} = f(P)$, then $L\{t^n F(t)\} = (-1)^n \frac{d^n}{dp^n}[f(P)]$

Proof: for $n = 1, 2, 3, \ldots$

Let $L\{F(t)\} = f(P)$ and

$$\frac{d^n}{dp^n} f(P) = f^n(P)$$

By definition of Laplace transform:

$$f(P) = \int_0^\infty e^{-pt} F(t)\,dt$$

Now

$$\frac{d}{dp} f(P) = \frac{d}{dp} \int_0^\infty e^{-pt} F(t)\,dt$$

$$\Rightarrow \frac{d}{dp} f(P) = \int_0^\infty -t\, e^{-pt} F(t)\,dt$$

$$\Rightarrow -\frac{d}{dP} f(P) = \int_0^\infty e^{-pt} \{t F(t)\}\,dt = L\{t F(t)\}$$

$$\therefore L\{t F(t)\} = -\frac{d}{dp} f(P)$$

$\therefore$ The theorem is true for $n=1$

Assume that the theorem is true for $n = r$

$$\therefore L\{t^r F(t)\} = (-1)^r \frac{d^r}{dp^r} f(P)$$

Now

$$(-1)^{r+1} \frac{d^{r+1}}{dp^{r+1}} f(P) = f^{(r+1)}(P)$$

$$= \int_0^\infty e^{-pt}\, t^{r+1} F(t)\,dt$$

$$= L\{t^{r+1} F(t)\}$$

$$\therefore L\{t^{r+1} F(t)\} = (-1)^{r+1} \frac{d^{r+1}}{dp^{r+1}} f(P)$$

$\therefore$ The theorem is true for $n = r+1$

By the principle of mathematical induction,

The theorem is true for $n \in N$.

Hence $$L\{t^n F(t)\} = (-1)^n \frac{d^n}{dp^n}[f(P)]$$

Problems based on multiplication by powers of t:

1. Find $L\{t \cos at\}$

Let $F(t) = \cos at$

Then $L\{F(t)\} = L\{\cos at\} = \frac{p}{p^2+a^2} = f(p)$ say

we know that

" If $L\{F(t)\} = f(p)$, then $L\{t^n F(t)\} = (-1)^n \frac{d^n}{dp^n} f(p)$ "

put $n=1$, $F(t) = \cos at$, $f(p) = \frac{p}{p^2+a^2}$

Then $L\{t \cos at\} = (-1) \frac{d}{dp} \frac{p}{p^2+a^2}$

$$= -1 \left\{ \frac{(p^2+a^2)(1) - p(2p)}{(p^2+a^2)^2} \right\}$$

$$= - \frac{\{p^2+a^2-2p^2\}}{(p^2+a^2)^2} = \frac{p^2-a^2}{(p^2+a^2)^2}$$

$$\therefore L\{t \cos at\} = \frac{p^2-a^2}{(p^2+a^2)^2}$$

2. Find $L\{t \sin t\}$

Let $F(t) = \sin t$

Then $L\{F(t)\} = L\{\sin t\} = \frac{1}{p^2+1} = f(p)$

we know that

" If $L\{F(t)\} = f(p)$, then $L\{t^n F(t)\} = (-1)^n \frac{d^n}{dp^n} f(p)$ "

put $n=1$, $F(t) = \sin t$, $f(p) = \frac{1}{p^2+1}$

Then $L\{t \sin t\} = (-1) \frac{d}{dp}\left(\frac{1}{p^2+1}\right)$

$$= -1 \left\{ \frac{(p^2+1)(0) - 1(2p)}{(p^2+1)^2} \right\}$$

$$= \frac{2p}{(p^2+1)^2}$$

$$\therefore L\{t \sin t\} = \frac{2p}{(p^2+1)^2}$$

3. Find $L\{t^2 e^{-at}\}$

Sol: Let $F(t) = e^{-at}$

Then $L\{F(t)\} = L\{e^{-at}\} = \frac{1}{p+a} = f(p)$

we know that

" If $L\{F(t)\} = f(p)$, then $L\{t^n F(t)\} = (-1)^n \frac{d^n}{dp^n} f(p)$ "

put $n=2$, $F(t) = e^{-at}$, $f(p) = \frac{1}{p+a}$

Then $L\{t^2 e^{-at}\} = \frac{d^2}{dp^2}\left(\frac{1}{p+a}\right)$

$$= \frac{d}{dp}\left[\frac{d}{dp}\left(\frac{1}{p+a}\right)\right]$$

$$= \frac{d}{dp}\left[\frac{-1}{(p+a)^2}\right]$$

$$= \frac{2}{(p+a)^3}$$

$$\boxed{\frac{d^2}{dt^2}\left(\frac{1}{t}\right) = \frac{2}{t^3}.}$$

$\therefore L\{t^2 e^{-at}\} = \frac{2}{(p+a)^3}$

4. Find $L\{t^2 \sin at\}$

Sol: Let $F(t) = \sin at$

Then $L\{F(t)\} = L\{\sin at\} = \frac{a}{p^2+a^2} = f(p)$

we know that:

If $L\{F(t)\} = f(p)$, Then $L\{t^n F(t)\} = (-1)^n \frac{d^n}{dp^n} f(p)$

put $n=2$, $F(t) = \sin at$, $f(p) = \frac{a}{p^2+a^2}$

Then $L\{t^2 \sin at\} = \frac{d^2}{dp^2}\left(\frac{a}{p^2+a^2}\right) = a\frac{d}{dp}\left[\frac{(p^2+a^2)(0) - 2p}{(p^2+a^2)^2}\right]$

$$= a\frac{d}{dp}\frac{-2p}{(p^2+a^2)^2}$$

$$= -2a\left\{\frac{(p^2+a^2)^2 1 - 2p(p^2+a^2)(2p)}{(p^2+a^2)^4}\right\}$$

$$= \frac{-2a\{a^2 - 3p^2\}}{(p^2+a^2)^3} = \text{Ans.}$$

5. Find i) $L\{te^{2t}\}$ ii) $L\{t^2 e^{2t}\}$

iii) $L\{t\cosh 3t\}$ iv) $L\{t^3 \cos t\}$ v) $L\{t \sinh at\}$

6. Find $L\{t[3\sin 2t - 2\cos 2t]\}$

Sol: Let $F(t) = 3\sin 2t - 2\cos 2t$

Then $L\{F(t)\} = L\{3\sin 2t - 2\cos 2t\}$

$$= 3L\{\sin 2t\} - 2L\{\cos 2t\}$$

$$= 3\frac{2}{p^2+4} - 2\cdot\frac{p}{p^2+4}$$

$$= \frac{6-2p}{p^2+4} = f(p)$$

we know that

If $L\{F(t)\} = f(p)$, then $L\{t^n F(t)\} = (-1)^n \frac{d^n}{dp^n} f(p)$

put $n=1$, $F(t) = 3\sin 2t - 2\cos 2t$, $f(p) = \frac{6-2p}{p^2+4}$

Then

$$L\{t[3\sin 2t - 2\cos 2t]\} = -\frac{d}{dp}\left[\frac{6-2p}{p^2+4}\right]$$

$$= -\left\{\frac{(p^2+4)(-2) - (6-2p)(2p)}{(p^2+4)^2}\right\}$$

$$= -\left\{\frac{-2p^2 - 8 - 12p + 4p^2}{(p^2+4)^2}\right\}$$

$$= \frac{12p + 8 - 2p^2}{(p^2+4)^2}$$

$$\therefore L\{t[3\sin 2t - 2\cos 2t]\} = \frac{12p+8-p^2}{(p^2+4)^2}$$

7. Find $L\{\sin at - at\cos at\}$

Sol: $L\{\sin at - at\cos at\} = L\{\sin at\} - aL\{t\cos at\}$

$$= \frac{a}{p^2+a^2} - a\left[-\frac{d}{dp} L(\cos at)\right]$$

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$$= \frac{a}{p^2+a^2} + a\frac{d}{dp}\frac{p}{p^2+a^2}$$

$$= \frac{a}{p^2+a^2} + a\left[\frac{p^2+a^2 - p(2p)}{(p^2+a^2)^2}\right]$$

$$= \frac{a}{p^2+a^2} + \frac{a(a^2-p^2)}{(p^2+a^2)^2}$$

$$= \frac{a}{p^2+a^2}\left[1 + \frac{a^2-p^2}{p^2+a^2}\right]$$

$$= \frac{a}{p^2+a^2}\left[\frac{2a^2}{p^2+a^2}\right]$$

$$= \frac{2a^3}{(p^2+a^2)^3}$$

$$\therefore L\{\sin at - at\cos at\} = \frac{2a^3}{(p^2+a^2)^3}$$

Note:

$$L\{tF(t)\} = -\frac{d}{dp} L\{F(t)\}$$

$$L\{t^2F(t)\} = \frac{d^2}{dp^2} L\{F(t)\}$$

$$L\{t^3F(t)\} = -\frac{d^3}{dp^3} L\{F(t)\}$$

8. Find $L\{t^n e^{at}\}$

Let $F(t) = e^{at}$

Then $L\{F(t)\} = L\{e^{at}\} = \frac{1}{p-a} = f(p)$

Now $L\{t^n e^{at}\} = (-1)^n \frac{d^n}{dp^n} f(p)$

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$$= (-1)^n \frac{d^n}{dp^n} \frac{1}{p-a}$$

$$= (-1)^n \frac{d^{n-1}}{dp^{n-1}} \left[\frac{-1}{(p-a)^2} \right]$$

$$= (-1)^n \frac{d^{n-2}}{dp^{n-2}} \left[\frac{2}{(p-a)^3} \right] \cdots\cdots$$

$$= (-1)^n (-1)^n \frac{n!}{(p-a)^{n+1}}$$

$$= \frac{n!}{(p-a)^{n+1}}$$ (here we choose n as even number}

$$\therefore L\{t^n e^{at}\} = \frac{n!}{(p-a)^{n+1}}$$

9. Find $L\{(t^2 - 3t + 2)\sin 3t\}$

Sol: Let $F(t) = \sin 3t$

Then $L\{F(t)\} = L\{\sin 3t\}$ $\qquad L\{\sin at\} = \frac{a}{p^2+a^2}$

$$= \frac{3}{p^2 + 3^2}$$

$$= \frac{3}{p^2+9} = f(p)$$

Now

$L\{(t^2 - 3t + 2)\sin 3t\} = L\{(t^2 - 3t + 2)F(t)\}$

$$= L\{t^2F(t) - 3tF(t) + 2F(t)\}$$

$$= L\{t^2F(t)\} - 3L\{tF(t)\} + 2L\{F(t)\}$$

$$= (-1)^2\frac{d^2}{dp^2}f(p) - 3(-1)\frac{d}{dp}f(p) + 2f(p)$$

$$= \frac{d^2}{dp^2}\left(\frac{3}{p^2+9}\right) + 3\frac{d}{dp}\frac{3}{p^2+9} + 2\cdot\frac{3}{p^2+9}$$

$$= 3\frac{d}{dp}\left[\frac{(p^2+9)(0)-(2p)}{(p^2+9)^2}\right] + 9\left[\frac{(p^2+9)(0)-2p}{(p^2+9)^2}\right] + \frac{6}{p^2+9}$$

$$= -6\frac{d}{dp}\left[\frac{p}{(p^2+9)^2}\right] + \frac{-18p}{(p^2+9)^2} + \frac{6}{p^2+9}$$

$$= -6\left\{\frac{(p^2+9)^2\cdot 1 - p\,2(p^2+9)(2p)}{(p^2+9)^4}\right\} + \frac{-18p+6(p^2+9)}{(p^2+9)^2}$$

$$= -6\left\{\frac{p^2+9-4p^2}{(p^2+9)^3}\right\} + \frac{-18p+6(p^2+9)}{(p^2+9)^2}$$

$$= -6\left\{\frac{9-3p^2}{(p^2+9)^3}\right\} - 6\left\{\frac{3p-2p^2-18}{(p^2+9)^2}\right\}$$

$$= \frac{18(p^2-3)}{(p^2+9)^3} - \frac{6(-2p^2+3p-18)}{(p^2+9)^2}$$

$$= \frac{18(p^2-3)}{(p^2+9)^3} - \frac{18p}{(p^2+9)^2} + \frac{6}{(p^2+9)}$$

= Ans.

Theorem: Division by t:

If $L\{F(t)\} = f(s)$, then $L\left\{\dfrac{F(t)}{t}\right\} = \int_s^{\infty} f(x)\,dx$,

Provided the integral exists.

Proof: Let $L\{F(t)\} = f(s)$

Then by definition of Laplace Transform:

$$f(s) = \int_0^{\infty} e^{-st} F(t)\,dt \longrightarrow ①$$

Integrating eq ① w.r.to s between s=s to s=∞

$$\int_s^{\infty} f(s)\,ds = \int_s^{\infty} ds \int_0^{\infty} e^{-st} F(t)\,dt$$

$$= \int_0^{\infty} \left(\int_s^{\infty} e^{-st}\,ds \right) F(t)\,dt$$

$$= \int_0^{\infty} \left[\frac{e^{-st}}{-t} \right]_{s=s}^{s=\infty} F(t)\,dt$$

$$= \int_0^{\infty} -\frac{1}{t}\left[e^{-\infty} - e^{-st} \right] F(t)\,dt$$

$$= \int_0^{\infty} -\frac{1}{t}\left[0 - e^{-st} \right] F(t)\,dt$$

$$= \int_0^{\infty} \left\{ \frac{F(t)}{t} \right\} e^{-st}\,dt$$

$$= L\left\{ \frac{F(t)}{t} \right\}$$

$$\therefore L\left\{ \frac{F(t)}{t} \right\} = \int_s^{\infty} f(s)\,ds$$

Hence proved.

P. Prove that $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left(\frac{1}{p}\right)$ and hence find $L\left\{\frac{\sin at}{t}\right\}$, thus Laplace transform of $\frac{\cos at}{t}$ does not exists

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Sol : Let $F(t) = \sin t$

Then $\lim_{t\to 0} \frac{F(t)}{t} = \lim_{t\to 0} \frac{\sin t}{t} = 1$

since $L\{F(t)\} = L\{\sin t\}$

$$= \frac{1}{p^2+1} = f(p)$$

Then by previous theorem:

$$L\left\{\frac{F(t)}{t}\right\} = \int_p^\infty f(x)\,dx$$

$$= \int_p^\infty \frac{1}{x^2+1}\,dx$$

$$= \left[\tan^{-1} x\right]_{x=p}^{\infty}$$

$$= \tan^{-1}(\infty) - \tan^{-1}(p)$$

$$= \frac{\pi}{2} - \tan^{-1}(p)$$

$$= \cot^{-1}(p)$$

$$= \tan^{-1}\left(\frac{1}{p}\right)$$

$$\therefore L\left\{\frac{F(t)}{t}\right\} = \tan^{-1}\left(\frac{1}{p}\right) \cancel{= f(p)}$$

i.e $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left(\frac{1}{p}\right) = f(p)$

Now $L\left\{\frac{\sin at}{t}\right\} = a\,L\left\{\frac{\sin at}{at}\right\}$

$$= a \cdot \frac{1}{a} f\left(\frac{p}{a}\right)$$

$$= \tan^{-1}\left(\frac{1}{(P/a)}\right)$$

$$= \tan^{-1}\left(\frac{a}{P}\right)$$

$$\therefore L\left\{\frac{\sin at}{t}\right\} = \tan^{-1}\left(\frac{a}{P}\right) = \cot^{-1}\left(\frac{P}{a}\right)$$ Ans

Now $L\{\cos at\} = \frac{P}{P^2+a^2} = f(P)$ say

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$$L\left\{\frac{\cos at}{t}\right\} = \int_P^\infty \frac{x}{x^2+a^2}\,dx$$

$$= \frac{1}{2}\int_P^\infty \frac{2x}{x^2+a^2}\,dx$$

$$= \frac{1}{2}\log\left[x^2+a^2\right]_{x=P}^{\infty}$$

This value does not exist

$\therefore L\left\{\frac{\cos at}{t}\right\}$ does not exists

P. show that $\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$

Let $F(t) = \sin t$

Then $\lim_{t\to 0} \frac{F(t)}{t} = \lim_{t\to 0} \frac{\sin t}{t} = 1$

since $L\{F(t)\} = L\{\sin t\}$

$$= \frac{1}{P^2+1} = f(P)$$

Then by previous theorem, we have

$$L\left\{\frac{F(t)}{t}\right\} = \int_P^\infty f(x)\,dx$$

$$= \int_P^\infty \frac{1}{x^2+1}\,dx$$

$$= \left[\tan^{-1} x\right]_{x=p}^{\infty}$$

$$= \tan^{-1}(\infty) - \tan^{-1}(p)$$

$$= \frac{\pi}{2} - \tan^{-1}(p) = \cot^{-1}(p) = \tan^{-1}\left(\frac{1}{p}\right)$$

$$\therefore L\left\{\frac{F(t)}{t}\right\} = \tan^{-1}\left(\frac{1}{p}\right)$$

i.e $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left(\frac{1}{p}\right)$

$$\Rightarrow \int_0^\infty e^{-pt} \frac{\sin t}{t}\, dt = \tan^{-1}\left(\frac{1}{p}\right)$$

Taking limit $p \to 0$ on both sides, we get

$$\lim_{p\to 0} \int_0^\infty e^{-pt}\left(\frac{\sin t}{t}\right) dt = \lim_{p\to 0} \tan^{-1}\left(\frac{1}{p}\right)$$

$$\Rightarrow \int_0^\infty \left[\lim_{p\to 0} e^{-pt}\right] \frac{\sin t}{t}\, dt = \tan^{-1}(\infty)$$

$$\Rightarrow \int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$$

$$\therefore \int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$$

Theorem: Initial Value Theorem:

If $L\{F(t)\} = f(s)$, then $\lim\limits_{t \to 0} F(t) = \lim\limits_{s \to \infty} s\, f(s)$

Proof:

Let $L\{F(t)\} = f(s)$

By previous theorem, we have

$$L\{F'(t)\} = s\, f(s) - \cancel{f(0)}\, F(0)$$

$$\Rightarrow \int_0^\infty e^{-st} F'(t)\, dt = s f(s) - F(0)$$

Taking limit $s \to \infty$ on both sides

$$\lim_{s \to \infty} \int_0^\infty e^{-st} F'(t)\, dt = \lim_{s \to \infty} \left[s f(s) - F(0) \right]$$

$$\Rightarrow \int_0^\infty \lim_{s \to \infty} e^{-st} F'(t)\, dt + F(0) = \lim_{s \to \infty} s f(s)$$

$$\Rightarrow \int_0^\infty e^{-\infty} F'(t)\, dt + \lim_{t \to 0} F(t) = \lim_{s \to \infty} s f(s)$$

$$\Rightarrow \int_0^\infty 0 \; F'(t)\, dt + \lim_{t \to 0} F(t) = \lim_{s \to \infty} s f(s)$$

$$\boxed{\therefore \lim_{t \to 0} F(t) = \lim_{s \to \infty} s f(s)}$$ Ans.

Theorem: Final Value Theorem:

If $L\{F(t)\} = f(s)$, then $\lim\limits_{t \to \infty} F(t) = \lim\limits_{s \to 0} s f(s)$

Proof:

Let $L\{F(t)\} = f(s)$

By previous Theorem, we have

$$L\{F'(t)\} = s f(s) - F(0)$$

$$\Rightarrow \int_0^{\infty} e^{-st} F'(t)\,dt = s f(s) - F(0)$$

Taking ~~s~~ limit $s \to 0$ on both sides

$$\lim_{s \to 0} \int_0^{\infty} e^{-st} F'(t)\,dt = \lim_{s \to 0} \left[s f(s) - F(0) \right]$$

$$\Rightarrow \int_0^{\infty} \left(\lim_{s \to 0} e^{-st} \right) F'(t)\,dt = \lim_{s \to 0} s f(s) - F(0)$$

$$\Rightarrow \int_0^{\infty} \frac{d}{dt} F(t)\,dt + F(0) = \lim_{s \to 0} s f(s)$$

$$\Rightarrow \left[F(t) \right]_0^{\infty} + F(0) = \lim_{s \to 0} s f(s)$$

$$\Rightarrow F(\infty) - F(0) + F(0) = \lim_{s \to 0} s f(s)$$

$$\Rightarrow \lim_{t \to \infty} F(t) = \lim_{s \to 0} s f(s)$$

$$\boxed{\therefore \lim_{t \to \infty} F(t) = \lim_{s \to 0} s f(s)}$$

Periodic Function:

A function $F(t)$ is said to be periodic function with period $T > 0$ if

$$T(u+nT) = F(u), \; \forall \; n = 1, 2, 3, \cdots$$

Theorem: Fundamental Theorem on periodic Functions:

Let $F(t)$ be a periodic function with period T. Then $L\{F(t)\} = \frac{1}{1-e^{-pT}} \int_0^T e^{-pt} F(t)\, dt$

Proof: Given that:

$F(t)$ is a periodic function with period T.

Then by definition of periodic function, we have

$$F(u+nT) = F(u), \text{ for } n = 1, 2, 3, \cdots$$

i.e $F(u+T) = F(u)$

$F(u+2T) = F(u)$

$F(u+3T) = F(u), \cdots$

By def of Laplace transform, we have

$$L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\, dt$$

$$= \int_0^T e^{-pt} F(t)\, dt + \int_T^{2T} e^{-pt} F(t)\, dt + \int_{2T}^{3T} e^{-pt} F(t)\, dt + \cdots + \cdots$$

$$= \int_0^T e^{-pt} F(t)\,dt + \int_0^T e^{-p(u+T)} F(u+T)\,du + \int_0^T e^{-p(u+2T)} F(u+2T)\,du + \cdots$$

$$= \int_0^T e^{-pt} F(t)\,dt + \int_0^T e^{-pu} e^{-pT} F(u)\,du + \int_0^T e^{-pu} e^{-p(2T)} F(u)\,du + \cdots$$

$$= \int_0^T e^{-pu} F(u)\,du + e^{-pT}\int_0^T e^{-pu} F(u)\,du + e^{-p(2T)}\int_0^T e^{-pu} F(u)\,du + \cdots$$

$$= \int_0^T e^{-pu} F(u)\,du \left[\underbrace{1 + e^{-pT} + e^{-2pT} + \cdots}\right]$$

$$(1-x)^{-1} = 1 + x + x^2 + \cdots \quad \text{where } x = e^{-pT}$$

$$= \int_0^T e^{-pu} F(u)\,du \left[\frac{1}{1-e^{-pT}}\right]$$

$$= \frac{1}{1-e^{-pT}} \int_0^T e^{-pt} F(t)\,dt$$

$$\therefore L\{F(t)\} = \frac{1}{1-e^{-pT}} \int_0^T e^{-pt} F(t)\,dt$$

Hence proved

P. If $F(t) = t^2,\ 0 < t < 2$ and $F(t+2) = F(t)$, Then find $L\{F(t)\}$.

<u>Sol</u>: Given that:

$F(t) = t^2,\ 0 < t < 2,\ F(t+2) = F(t)$

here $F(t)$ is periodic function with period 2

Then

$$L\{F(t)\} = \frac{1}{1-e^{-pT}} \int_0^T e^{-pt} F(t)\,dt$$

$$= \frac{1}{1-e^{-p2}} \int_0^2 e^{-pt} t^2\,dt$$

$$= \frac{1}{1-e^{-2p}} \left\{ t^2 \frac{e^{-pt}}{-p} + \int_0^2 2t \frac{e^{-pt}}{p}\,dt \right\}$$

$$= \frac{1}{1-e^{-2p}} \left\{ \frac{t^2 e^{-pt}}{-p} + \frac{2}{p}\left[\frac{t e^{-pt}}{-p} - \frac{e^{-pt}}{p^2}\right] \right\}_{t=0}^{t=2}$$

$$= \frac{1}{1-e^{-2p}} \left\{ \frac{4e^{-2p}}{-p} + \frac{2}{p}\left[\frac{2e^{-p2}}{-p} - \frac{e^{-2p}}{p^2}\right] - \left[\frac{2}{p}\left(-\frac{1}{p^2}\right)\right] \right\}$$

$$= \frac{1}{1-e^{-2p}} \left\{ -\frac{4}{p} e^{-2p} - \frac{4}{p^2} e^{-2p} - \frac{2}{p^3} e^{-2p} + \frac{2}{p^3} \right\}$$

$$= \frac{1}{1-e^{-2p}} \left\{ \frac{2 - 4p^2 e^{-2p} - 4p e^{-2p} - 2e^{-2p}}{p^3} \right\}$$

$$\therefore L\{F(t)\} = \frac{1}{1-e^{-2p}} \left\{ \frac{2 - e^{-2p}[4p^2 + 4p + 2]}{p^3} \right\}$$

$$\int f(x) g(x)\,dx = f(x) \int g(x)\,dx - \int f'(x) \int g(x)\,dx$$

$f(x) = t^2$, $g(x) = e^{-pt}$

$$= t^2\left(\frac{e^{-pt}}{-p}\right) - \int 2t\left(\frac{e^{-pt}}{-p}\right) dt$$

Note : $\because$ $F(t)$ is periodic with period T

$$\int_T^{2T} e^{-pt} F(t)\,dt = \int_0^T e^{-p(u+T)} F(u+T)\,du$$

$$= \int_0^T e^{-pu} e^{-pT} F(u)\,du$$

$$= e^{-pT} \int_0^T e^{-pu} F(u)\,du$$

$$= e^{-pT} \int_0^T e^{-pt} F(t)\,dt$$

$$\int_{2T}^{3T} e^{-pT} F(t)\,dt = \int_0^T e^{-p(u+2T)} F(u+2T)\,du$$

$$= \int_0^T e^{-pu} e^{-p(2T)} F(u)\,du$$

$$= e^{-2pT} \int_0^T e^{-pu} F(u)\,du$$

$$= e^{-2pT} \int_0^T e^{-pt} F(t)\,dt$$

Theorem: Existance of Laplace Transform:

If $F(t)$ is a function of class A, Then $L\{F(t)\}$ exists

(OR)

Suppose $F(t)$ is piecewise continuous in every finite interval and is of exponential order "a" as $t \to \infty$, Then Laplace transform of $F(t)$ exists.

Proof:

Let $F(t)$ is a function of class A.

Then $F(t)$ is piecewise continuous in every finite interval and

$F(t)$ is of exponential order 'a' as $t \to \infty$

we have to prove that:

$$L\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\,dt \text{ exists}$$

For any $t_0 > 0$:

$$\int_0^\infty e^{-st} F(t)\,dt = \int_0^{t_0} e^{-st} F(t)\,dt + \int_{t_0}^\infty e^{-st} F(t)\,dt \to (1)$$

since $F(t)$ is piecewise continuous.

Then $\int_0^{t_0} e^{-st} \cdot F(t)\,dt$ exists $\to (2)$

Since $F(t)$ is of exponential order a as $t \to \infty$

$\Rightarrow \lim_{t \to \infty} e^{-at} F(t)$ is finite

$\Rightarrow \exists$ a +ve integer m and a +ve real number M s.t

$$|e^{-at} F(t)| < M, \forall\ t \geq m$$

Now

$$\left|\int_{t_0}^{\infty} e^{-st} F(t)\,dt\right| \le \int_{t_0}^{\infty} \left|e^{-st} F(t)\right| dt$$

$$\le \int_{t_0}^{\infty} e^{-st} |F(t)|\,dt$$

$$\le \int_{t_0}^{\infty} e^{-st} M e^{at}\,dt$$

$$\le M \int_{t_0}^{\infty} e^{-(s-a)t}\,dt$$

$$\le M \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_{t=t_0}^{\infty}$$

$$\le -\frac{M}{s-a}\left[e^{-\infty} - e^{-(s-a)t_0}\right]$$

$$\le \frac{M}{s-a} e^{-(s-a)t_0}$$

$$\therefore \left|\int_{t_0}^{\infty} e^{-st} F(t)\,dt\right| \le \frac{M}{s-a} e^{-(s-a)t_0} \longrightarrow (3)$$

For large values of t_0 : $\frac{M}{s-a} e^{-(s-a)t_0}$ is very small.

$\therefore$ from eq (3) : $\int_{t_0}^{\infty} e^{-st} F(t)\,dt$ exists $\longrightarrow$ (4)

From eq's (2) and (4) :

$$\int_{0}^{\infty} e^{-st} F(t)\,dt \text{ exists}$$

P. Find Laplace Transform of $L\{F(t)\}$.

Sol :

$$L\{L[F(t)]\} = L\left\{\int_0^\infty e^{-st} F(t)\,dt\right\}$$

$$= L\{G(t)\} \; ; \; G(t) = \int_0^\infty e^{-st} F(t)\,dt$$

$$= \int_0^\infty e^{-su}\, G(t)\, ds$$

$$= \int_0^\infty e^{-su}\, ds \int_0^\infty e^{-st} F(t)\,dt$$

$$= \int_0^\infty F(t)\,dt \int_0^\infty e^{-s(t+u)}\, ds$$

$$= \int_0^\infty F(t)\,dt \left[\frac{e^{-s(t+u)}}{-(t+u)}\right]_0^\infty$$

$$= \int_0^\infty \frac{F(t)}{-(t+u)}\,dt \left[e^{-\infty} - e^{0}\right]$$

$$= \int_0^\infty \frac{F(t)}{t+u}\,dt$$

$$\therefore L\{L[F(t)]\} = \int_0^\infty \frac{F(t)}{t+u}\,dt$$

Note:

1. $e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$

2. $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^n}{n!} + \cdots$

3. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

4. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

5. $\int \frac{1}{x^2+1}\, dx = \tan^{-1} x$

6. $\int \frac{f'(x)}{f(x)}\, dx = \log [f(x)]$

7. $\tan^{-1} x + \cot^{-1} x = \cancel{\text{to}}\ \frac{\pi}{2}$

8. $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

9. $L\{t^n\} = \frac{\Gamma(n+1)}{p^{n+1}}$

10. $\Gamma(n+1) = n\Gamma(n)$

11. $\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$

12. $\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$

13. $\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$

14. $\Gamma\left(\frac{9}{2}\right) = \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$

P. Find Laplace Transform of $\sin(\sqrt{t})$

Sol:

Let $F(t) = \sin(\sqrt{t})$

$$= \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \frac{(\sqrt{t})^7}{7!} + \cdots$$

Now

$$L\{F(t)\} = L\{\sin\sqrt{t}\}$$

$$= L\left\{\sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \frac{(\sqrt{t})^7}{7!} + \cdots\right\}$$

$$= L\{t^{\frac{1}{2}}\} - \frac{1}{3!}L\{t^{3/2}\} + \frac{1}{5!}L\{t^{5/2}\} - \frac{1}{7!}L\{t^{7/2}\} + \cdots$$

$$= \frac{\Gamma(\frac{1}{2}+1)}{p^{\frac{1}{2}+1}} - \frac{1}{3!}\frac{\Gamma(\frac{3}{2}+1)}{p^{3/2+1}} + \frac{1}{5!}\frac{\Gamma(\frac{5}{2}+1)}{p^{5/2+1}} - \cdots$$

$$= \frac{\frac{1}{2}\Gamma(\frac{1}{2})}{p^{3/2}} - \frac{1}{3!}\frac{\frac{3}{2}\Gamma(\frac{3}{2})}{p^{5/2}} + \frac{1}{5!}\frac{\frac{5}{2}\Gamma(\frac{5}{2})}{p^{7/2}} - \cdots$$

$$= \frac{1}{p^{3/2}}\left\{\frac{1}{2}\sqrt{\pi} - \frac{\frac{1}{3!}\frac{3}{2}\frac{1}{2}\sqrt{\pi}}{p} + \frac{1}{5!}\frac{\frac{5}{2}\cdot\frac{3}{2}\frac{1}{2}\sqrt{\pi}}{p^2} - \cdots\right\}$$

$$= \frac{\sqrt{\pi}}{2p^{3/2}}\left\{1 - \frac{1}{4p} + \frac{1}{32p^2} - \cdots\right\}$$

$$= \frac{\sqrt{\pi}}{2p^{3/2}}\left\{1 - \frac{(1/4p)}{1!} + \frac{\left(\frac{1}{4p}\right)^2}{2!} - \cdots\right\}$$

$$= \frac{\sqrt{\pi}}{2p^{3/2}} e^{-\frac{1}{4p}}$$

$$\therefore L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2p^{3/2}} e^{-\frac{1}{4p}}$$

P. show that $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{p}}\, e^{-\frac{1}{4p}}$

Sol:

Let $F(t) = \sin\sqrt{t} \Rightarrow F(0) = 0$

Then $F'(t) = \frac{d}{dt} F(t)$

$$= \frac{d}{dt}[\sin(\sqrt{t})]$$

$$= \frac{\cos\sqrt{t}}{2\sqrt{t}}$$

Now $L\{F(t)\} = L\{\sin\sqrt{t}\}$

$$= \frac{\sqrt{\pi}}{2p^{3/2}} e^{-\frac{1}{4p}} = f(p)$$

We know that

$$L\{F'(t)\} = p f(p) - F(0)$$

$$\Rightarrow L\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = \frac{p\sqrt{\pi}}{2p^{3/2}} e^{-\frac{1}{4p}} - 0$$

$$\Rightarrow \frac{1}{2} L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{2p^{1/2}} e^{-\frac{1}{4p}}$$

$$\Rightarrow L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{p}}\, e^{-\frac{1}{4p}}$$

$$\therefore L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{p}}\, e^{-\frac{1}{4p}}$$

Evaluation of Integrals:

If $L\{F(t)\} = f(p)$

i.e $\int_0^\infty e^{-pt} F(t)\,dt = f(p)$

Taking limit $p \to 0$ we get

$$\int_0^\infty F(t)\,dt = f(0)$$

1. Evaluate $\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} \, dt$

Sol:

Let $F(t) = e^{-at} - e^{-bt}$

$$L\{F(t)\} = L\{e^{-at} - e^{-bt}\}$$

$$= L\{e^{-at}\} - L\{e^{-bt}\}$$

$$= \frac{1}{p+a} - \frac{1}{p+b} = f(p)$$

We know that

$$L\left\{\frac{F(t)}{t}\right\} = \int_p^\infty f(x)\,dx$$

$$= \int_p^\infty \left[\frac{1}{x+a} - \frac{1}{x+b}\right] dx$$

$$= \left[\log\left(\frac{x+a}{x+b}\right)\right]_p^\infty$$

$$= 0 - \log\left(\frac{p+a}{p+b}\right)$$

$$\therefore L\left\{\frac{F(t)}{t}\right\} = \log\left[\frac{p+b}{p+a}\right]$$

$$\Rightarrow \int_0^\infty e^{-pt} \frac{F(t)}{t} \, dt = \log\left[\frac{p+b}{p+a}\right]$$

Taking limit $p \to 0$ on both sides

$$\lim_{p \to 0} \int_0^\infty e^{-pt} \frac{F(t)}{t} \, dt = \lim_{p \to 0} \left[\frac{p+b}{p+a}\right]$$

$$\Rightarrow \int_0^\infty e^{0}\left(\frac{e^{-at} - e^{-bt}}{t}\right) dt = \log\left(\frac{0+b}{0+a}\right)$$

Hence $\boxed{\int_0^\infty \left[\frac{e^{-at} - e^{-bt}}{t}\right] dt = \log\left(\frac{b}{a}\right)}$

Imp

2. S.T $\int_0^\infty \frac{e^{-3t} - e^{-2t}}{t} dt = \log\left(\frac{2}{3}\right)$

We know that

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt = \log\left(\frac{b}{a}\right)$$

put $a=3,\ b=2$

$$\int_0^\infty \frac{e^{-3t} - e^{-2t}}{t} dt = \log\left(\frac{2}{3}\right)$$

3. Evaluate $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t} dt$

We know that

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt = \log\left(\frac{b}{a}\right)$$

put $b=3,\ a=1$

$$\int_0^\infty \frac{e^{-t} - e^{-3t}}{t} dt = \log(3)$$

4. Show that $\int_0^\infty t\, e^{-3t} \sin t\, dt = \frac{3}{50}$

Sol: $\int_0^\infty t\, e^{-pt} \sin t\, dt = \int_0^\infty e^{-pt} [t \sin t]\, dt$

$$= L\{t \sin t\}$$

$$= (-1)^1 \frac{d}{dp} L\{\sin t\}$$

$$= -\frac{d}{dp}\left[\frac{1}{p^2+1}\right]$$

$$= -\left[\frac{(p^2+1)(0) - 2p}{(p^2+1)^2}\right]$$

$$= \frac{2p}{(p^2+1)^2}$$

$$\therefore \int_0^\infty e^{-pt}(t \sin t)\, dt = \frac{2p}{(p^2+1)^2}$$

put $p = 3$

$$\int_0^\infty e^{-3t}\, t \sin t\, dt = \frac{6}{(10)^2} = \frac{3}{50}$$

5. Evaluate $\int_0^\infty t\, e^{-2t} \cos t\, dt$

$$\int_0^\infty t\, e^{-pt} \cos t\, dt = \int_0^\infty e^{-pt}[t \cos t]\, dt$$

$$= L\{t \cos t\} = (-1)\frac{d}{dp} L\{\cos t\}$$

$$= -\frac{d}{dp}\left[\frac{p}{p^2+1}\right] =$$

$$= -\left\{\frac{(p^2+1)\,1 - p(2p)}{(p^2+1)^2}\right\} = \frac{p^2-1}{(p^2+1)^2}$$

$$\therefore \int_0^\infty t\, e^{-pt} \cos t\, dt = \frac{p^2-1}{(p^2+1)^2}$$

put $p = 2$

$$\int_0^\infty t\, e^{-2t} \cos t\, dt = \frac{3}{25}$$

6. S.T $\int_0^\infty t^3 e^{-t} \sin t \, dt = 0$

Sol:

$$\int_0^\infty t^3 e^{-pt} \sin t \, dt = \int_0^\infty e^{-pt} [t^3 \sin t] \, dt$$

$$= L\{t^3 \sin t\}$$

$$= (-1)^3 \frac{d^3}{dp^3} L\{\sin t\}$$

$$= -\frac{d^3}{dp^3}\left[\frac{1}{p^2+1}\right]$$

$$= -\frac{d^2}{dp^2}\left[\frac{d}{dp}\left(\frac{1}{p^2+1}\right)\right]$$

$$= -\frac{d^2}{dp^2}\left[\frac{(p^2+1)0 - 2p}{(p^2+1)^2}\right]$$

$$= \frac{d^2}{dp^2}\left[\frac{2p}{(p^2+1)^2}\right] = \frac{d}{dp}\left[\frac{(p^2+1)^2(2) - 2(2p)(p^2+1)(2p)}{(p^2+1)^4}\right]$$

$$= \frac{d}{dp}\left[\frac{2(p^2+1) - 8p^2}{(p^2+1)^3}\right] = \frac{d}{dp}\left[\frac{2-6p^2}{(p^2+1)^3}\right]$$

$$= \frac{(p^2+1)^3[-12p] - [2-6p^2]\,3(p^2+1)^2(2p)}{(p^2+1)^6}$$

$$= \frac{(p^2+1)^2\{(p^2+1)(-12p) - 6p(2-6p^2)\}}{(p^2+1)^6}$$

$$= \frac{-12p^3 - 12p - 12p + 36p^3}{(p^2+1)^4}$$

$$= \frac{24p^3 - 24p}{(p^2+1)^4}$$

$$\therefore \int_0^\infty t^3 e^{-pt} \sin t \, dt = \frac{24p^3 - 24p}{(p^2+1)^4}$$

put $p = 1$

$$\int_0^\infty t^3 e^{-t} \sin t \, dt = \frac{24 - 24}{(1+1)^4} = 0$$

$$\therefore \int_0^\infty t^3 e^{-t} \sin t \, dt = 0$$

$\rightarrow\!\!\times\!\!-$

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

Bessel Function of order n:

Bessel function of order n is denoted by $J_n(x)$ and defined as

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)}\left\{1 - \frac{x^2}{2\cdot(2n+2)} + \frac{x^4}{2\cdot 4\cdot(2n+2)(2n+4)} - \cdots\right\}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

1. Find $L\{t\,J_0(at)\}$

Sol: By def of Bessel function, we have

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)}\left\{1 - \frac{x^2}{2\cdot(2n+2)} + \frac{x^4}{2\cdot 4\cdot(2n+2)(2n+4)} - \cdots\right\}$$

put $n=0$, $x=t$

$$J_0(t) = \frac{t^0}{2^0\Gamma(1)}\left\{1 - \frac{t^2}{2\cdot 2} + \frac{t^4}{2\cdot 4\cdot 2\cdot 4} - \cdots\right\}$$

$$= 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2\cdot 4^2} - \cdots$$

Taking Laplace Transforms on both sides, we get

$$L\{J_0(t)\} = L\left\{1 - \frac{t^2}{2^2} + \frac{t^4}{2^2\cdot 4^2} - \cdots\right\}$$

$$= L\{1\} - \frac{1}{2^2}L\{t^2\} + \frac{1}{2^2\cdot 4^2}L\{t^4\} - \cdots$$

$$= \frac{1}{p} - \frac{1}{2^2}\cdot\frac{2!}{p^3} + \frac{1}{2^2\cdot 4^2}\,\frac{4!}{p^5} - \cdots$$

$$= \frac{1}{p}\left\{1 - \frac{1}{2}\left(\frac{1}{p^2}\right) + \frac{3}{8}\left(\frac{1}{p^2}\right)^2 - \cdots\right\}$$

$$= \frac{1}{p}\left\{1 - \frac{\left(\frac{1}{2}\right)}{1!}\left(\frac{1}{p^2}\right) + \frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)}{2!}\left(\frac{1}{p^2}\right)^2 - \cdots\right\}$$

$$= \frac{1}{p}\left\{1 + \frac{1}{p^2}\right\}^{-\frac{1}{2}}$$

$$= \frac{1}{p}\left\{\frac{p^2+1}{p^2}\right\}^{-\frac{1}{2}}$$

$$= \frac{1}{p}\left\{\frac{p^2}{p^2+1}\right\}^{\frac{1}{2}}$$

$$= \frac{1}{p}\frac{\sqrt{p^2}}{\sqrt{p^2+1}}$$

$$= \frac{1}{p}\frac{p}{\sqrt{p^2+1}} = \frac{1}{\sqrt{p^2+1}}$$

$$\therefore L\{J_0(t)\} = \frac{1}{\sqrt{p^2+1}} = f(p)$$

By change of scale property, we have

If $L\{F(t)\} = f(p)$, then $L\{F(at)\} = \frac{1}{a}f\left(\frac{p}{a}\right)$

Then:

if $L\{J_0(t)\} = f(p)$, then $L\{J_0(at)\} = \frac{1}{a}f\left(\frac{p}{a}\right)$

$$\therefore L\{J_0(at)\} = \frac{1}{a}\frac{1}{\sqrt{\left(\frac{p}{a}\right)^2+1}}$$

$$= \frac{1}{a}\frac{1}{\sqrt{\frac{p^2+a^2}{a^2}}}$$

$$= \frac{1}{a}\frac{a}{\sqrt{p^2+a^2}} = \frac{1}{\sqrt{p^2+a^2}}$$

$$\therefore L\{J_0(at)\} = \frac{1}{\sqrt{p^2+a^2}}$$

We know that :

$$L\{t^n F(t)\} = (-1)^n \frac{d^n}{dp^n} L\{F(t)\}$$

put $n=1$, $F(t) = J_0(at)$

$$L\{t J_0(at)\} = -\frac{d}{dp} L\{J_0(at)\}$$

$$= -\frac{d}{dp}\left\{\frac{1}{\sqrt{p^2+a^2}}\right\}$$

$$= -\left\{\frac{\sqrt{p^2+a^2}\,(0) - \frac{1}{2\sqrt{p^2+a^2}}(2p)}{(\sqrt{p^2+a^2})^2}\right\}$$

$$= \frac{p}{\sqrt{p^2+a^2}\,(p^2+a^2)}$$

$$= \frac{p}{(p^2+a^2)^{3/2}}$$

$$\therefore L\{t J_0(at)\} = \frac{p}{(p^2+a^2)^{3/2}}$$

2. Prove that $L\{J_0(at)\} = \frac{1}{\sqrt{p^2+a^2}}$

Sol: We know that:

$$L\{J_0(t)\} = \frac{1}{\sqrt{p^2+1}} = f(p)$$

By change of scale property

If $L\{F(t)\} = f(p)$, then $L\{F(at)\} = \frac{1}{a} f\left(\frac{p}{a}\right)$

lly If $L\{J_0(t)\} = f(p)$, Then $L\{J_0(at)\} = \frac{1}{a} f\left(\frac{p}{a}\right)$

$$\therefore L\{J_0(at)\} = \frac{1}{a}\,\frac{1}{\sqrt{\left(\frac{p}{a}\right)^2+1}}$$

$$= \frac{1}{a}\,\frac{1}{\frac{\sqrt{p^2+a^2}}{a}} = \frac{1}{\sqrt{p^2+a^2}}$$

$$\therefore L\{J_0(at)\} = \frac{1}{\sqrt{p^2+a^2}}$$

3. Prove that $L\{e^{-at} J_0(at)\} = \frac{1}{\sqrt{p^2+2ap+2a^2}}$

Sol :

we know that

$$L\{J_0(at)\} = \frac{1}{\sqrt{p^2+a^2}} = f(p)$$

By First shifting Theorem, we have

If $L\{F(t)\} = f(p)$, then $L\{e^{-at} F(t)\} = f(p+a)$

put $F(t) = J_0(at)$

Then $L\{e^{-at} J_0(at)\} = \frac{1}{\sqrt{(p+a)^2+a^2}}$

$$= \frac{1}{\sqrt{p^2+2ap+2a^2}} \text{ Ans.}$$

4. Prove that $\int_0^\infty J_0(t)\, dt = 1$

Sol : we know that :

$$L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\, dt$$

put $F(t) = J_0(t)$

$$L\{J_0(t)\} = \int_0^\infty e^{-pt} J_0(t)\, dt$$

$$\Rightarrow \frac{1}{\sqrt{p^2+1}} = \int_0^\infty e^{-pt} J_0(t)\, dt$$

put $p=0$, we get

$$\frac{1}{\sqrt{0+1}} = \int_0^\infty e^{0} J_0(t)\, dt$$

$$\Rightarrow 1 = \int_0^\infty J_0(t)\, dt$$

$$\therefore \int_0^\infty J_0(t)\, dt = 1$$

5. Prove that $L\{J_1(t)\} = 1 - \frac{p}{\sqrt{p^2+1}}$ and hence deduce that $L\{t J_1(t)\} = \frac{1}{(p^2+1)^{3/2}}$.

Sol:

we know that:

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \left\{ 1 - \frac{x^2}{2\cdot(2n+2)} + \frac{x^4}{2\cdot 4\cdot(2n+2)(2n+4)} - \cdots \right\}$$

put $n=1,\ x=t$

$$J_1(t) = \frac{t}{2\Gamma(2)} \left\{ 1 - \frac{t^2}{2\cdot 4} + \frac{t^4}{2\cdot 4\cdot 4\cdot 6} - \cdots \right\}$$

$$= \frac{t}{2} - \frac{t^3}{2^2\cdot 4} + \frac{t^5}{2^2\cdot 4^2\cdot 6} - \cdots$$

Taking Laplace Transform on both sides, we get

$$L\{J_1(t)\} = L\left\{ \frac{t}{2} - \frac{t^3}{2^2\cdot 4} + \frac{t^5}{2^2\cdot 4^2\cdot 6} - \cdots \right\}$$

$$= \frac{1}{2} L\{t\} - \frac{1}{2^2\cdot 4} L\{t^3\} + \frac{1}{2^2\cdot 4^2\cdot 6} L\{t^5\} - \cdots$$

$$= \frac{1}{2}\,\frac{1}{p^2} - \frac{1}{2^2\cdot 4}\,\frac{3!}{p^4} + \frac{1}{2^2\cdot 4^2\cdot 6}\cdot\frac{5!}{p^6} - \cdots$$

$$= \frac{\left(\frac{1}{2}\right)\left(\frac{1}{p^2}\right)}{1!} - \frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)}{2!}\left(\frac{1}{p^2}\right)^2 + \frac{\left(\frac{1}{2}\right)\left(\frac{3}{5}\right)\left(\frac{5}{2}\right)}{3!}\left(\frac{1}{p^2}\right)^3 - \cdots$$

$$= 1 - \left\{ 1 - \frac{\left(\frac{1}{2}\right)}{1!}\left(\frac{1}{p^2}\right) + \frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)}{2!}\left(\frac{1}{p^2}\right)^2 - \cdots \right\}$$

$$= 1 - \left\{ 1 + \frac{1}{p^2} \right\}^{-\frac{1}{2}}$$

$$= 1 - \left\{ \frac{p^2+1}{p^2} \right\}^{-\frac{1}{2}}$$

$$= 1 - \left\{ \frac{p^2}{p^2+1} \right\}^{\frac{1}{2}}$$

$$= 1 - \frac{p}{\sqrt{p^2+1}}$$

$$\therefore L\{J_1(t)\} = 1 - \frac{p}{\sqrt{p^2+1}} = f(p)$$

we know that:

$$L\{t\,F(t)\} = -\frac{d}{dp} L\{F(t)\}$$

replace $F(t)$ by $J_1(t)$

$$L\{t\,J_1(t)\} = -\frac{d}{dp} L\{J_1(t)\}$$

$$= -\frac{d}{dp}\left[1 - \frac{p}{\sqrt{p^2+1}}\right\}$$

$$= \frac{d}{dp}\left[\frac{p}{\sqrt{p^2+1}}\right]$$

$$= \frac{\sqrt{(p^2+1)}\cdot 1 - p\,\frac{1}{2\sqrt{p^2+1}}\,2p}{p^2+1}$$

$$= \frac{\sqrt{p^2+1}}{p^2+1} - \frac{p^2}{(p^2+1)\sqrt{p^2+1}}$$

$$= \frac{p^2+1-p^2}{(p^2+1)\sqrt{p^2+1}}$$

$$= \frac{1}{(p^2+1)^{3/2}}$$

$$\therefore L\{t\,J_1(t)\} = \frac{1}{(p^2+1)^{3/2}}$$

$= \text{Ans.}$

The unit Step Function (OR) Heaviside's Unit Function:

The Heaviside unit function is denoted by $H(t)$ and defined as

$$H(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \quad \text{(OR)}$$

$$H(t-a) = \begin{cases} 1 & \text{if } t \geq a \\ 0 & \text{if } t < a \end{cases}$$

P. Find $L\{F(t-a)\}$, where $F(t-a)$ is Heaviside unit step function.

Sol: By definition of Heaviside unit step function:

$$F(t-a) = \begin{cases} 1 & \text{if } t \geq a \\ 0 & \text{if } t < a \end{cases}$$

By definition of Laplace Transform, we have

$$L\{F(t-a)\} = \int_0^\infty e^{-pt} F(t-a)\, dt$$

$$= \int_0^a e^{-pt} F(t-a)\,dt + \int_a^\infty e^{-pt} F(t-a)\,dt$$

$$= \int_0^a e^{-pt} \cdot 0 \, dt + \int_a^\infty e^{-pt} \cdot 1 \, dt$$

$$= \left[\frac{e^{-pt}}{-p}\right]_a^\infty$$

$$= -\frac{1}{p}\left[e^{-\infty} - e^{-pa}\right]$$

$$= \frac{1}{p} e^{-pa}$$

$$\therefore L\{F(t-a)\} = \frac{1}{p} e^{-pa}$$

Sine Integral:

The sine integral is denoted by $S_i(t)$ and defined as $\int_0^t \frac{\sin u}{u}\,du$

i.e $S_i(t) = \int_0^t \frac{\sin u}{u}\,du$

P. Find the L.T of $S_i(t)$.

Sol: we know that

$$S_i(t) = \int_0^t \frac{\sin u}{u}\,du$$

$$= \int_0^t \frac{1}{u}\left[u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots\right]du$$

$$= \int_0^t \left[1 - \frac{u^2}{3!} + \frac{u^4}{5!} - \cdots\right]du$$

$$= \left[u - \frac{u^3}{3\cdot 3!} + \frac{u^5}{5\cdot 5!} - \cdots\right]_0^t$$

$$\therefore S_i(t) = t - \frac{t^3}{3\cdot 3!} + \frac{t^5}{5\cdot 5!} - \cdots$$

Taking L.T on both sides, we get

$$L\{S_i(t)\} = L\left\{t - \frac{t^3}{3\cdot 3!} + \frac{t^5}{5\cdot 5!} - \cdots\right\}$$

$$= L\{t\} - \frac{1}{3\cdot 3!}L\{t^3\} + \frac{1}{5\cdot 5!}L\{t^5\} - \cdots$$

$$= \frac{1}{p^2} - \frac{1}{3\cdot 3!}\cdot\frac{3!}{p^4} + \frac{1}{5\cdot 5!}\,\frac{5!}{p^6} - \cdots$$

$$= \frac{1}{p}\left[\frac{1}{p} - \frac{1}{3p^3} + \frac{1}{5p^5} - \cdots\right]$$

$$= \frac{1}{p}\tan^{-1}\left(\frac{1}{p}\right)$$

$$\therefore L\{S_i(t)\} = \frac{1}{p}\tan^{-1}\left(\frac{1}{p}\right)$$

Cosine Integral:

The cosine integral is denoted by $C_i(t)$ and defined as $\int_t^{\infty} \frac{\cos u}{u}\,du$ i.e $\boxed{C_i(t) = \int_t^{\infty} \frac{\cos u}{u}\,du}$

P. Find L.T of $C_i(t)$

Sol: Let $F(t) = C_i(t) = \int_t^{\infty} \frac{\cos u}{u}\,du$

$$= -\int_{\infty}^{t} \frac{\cos u}{u}\,du$$

Now $F'(t) = \frac{d}{du}[F(t)] = -\frac{d}{du}\int_{\infty}^{t} \frac{\cos u}{u}\,du$

$$= -\frac{\cos t}{t} \qquad \left(\because \lim_{t\to\infty} F(t) = 0\right)$$

$$\therefore F'(t) = -\frac{\cos t}{t}$$

$$\Rightarrow t\,F'(t) = -\cos t$$

$$\Rightarrow L\{t\,F'(t)\} = -L\{\cos t\}$$

$$\Rightarrow -\frac{d}{dp} L\{F'(t)\} = -\frac{p}{p^2+1}$$

$$\Rightarrow \frac{d}{dp} L\{F'(t)\} = \frac{p}{p^2+1}$$

$$\Rightarrow \frac{d}{dp}\{p f(p) - F(0)\} = \frac{p}{p^2+1}$$

$$\Rightarrow \frac{d}{dp}\{p\,f(p)\} = \frac{p}{p^2+1}$$

on integrating both sides, we get

$$\int \frac{d}{dp}\{p\,f(p)\}\,dp = \int \frac{p}{p^2+1}\,dp$$

$$\Rightarrow p\,f(p) = \frac{1}{2}\int \frac{2p}{p^2+1}\,dp$$

$\Rightarrow \quad p f(p) = \frac{1}{2} \log(p^2+1) + c \quad \longrightarrow (1)$

Taking limit $p \to 0$, we get

$$\lim_{p\to 0} p f(p) = \frac{1}{2} \lim_{p\to 0} \log(p^2+1) + c$$

$\Rightarrow \quad 0 = \frac{1}{2}(0) + c$

$\Rightarrow \quad c = 0$

Then eq (1) $\Rightarrow \quad p f(p) = \frac{1}{2} \log(p^2+1)$

$\Rightarrow \quad p L\{F(t)\} = \frac{1}{2} \log(p^2+1)$

$\Rightarrow \quad L\{F(t)\} = \frac{1}{2p} \log(p^2+1)$

$\Rightarrow \quad L\{c_i(t)\} = \frac{1}{2p} \log(p^2+1)$

$$\therefore L\{c_i(t)\} = L\left\{\int_t^{\infty} \frac{\cos u}{u}\, du\right\} = \frac{1}{2p} \log(p^2+1)$$

The Error Function:

The error function is denoted by $\text{erf}(\sqrt{t})$ and defined as $\frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-\mu^2} d\mu$

i.e $$\boxed{\text{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-\mu^2} d\mu}$$

P. Find Laplace Transform of $\text{erf}(\sqrt{t})$ and hence prove that $L\{t\, \text{erf}(2\sqrt{t})\} = \frac{3p+8}{p^2(p+4)^{3/2}}$

Sol: we know that

$$\text{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-u^2} du$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} \left[1 - \frac{u^2}{1!} + \frac{(u^2)^2}{2!} - \frac{(u^2)^3}{3!} + \cdots\right] du$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} \left[1 - u^2 + \frac{u^4}{2} - \frac{u^6}{6} + \cdots\right] du$$

$$= \frac{2}{\sqrt{\pi}} \left[u - \frac{u^3}{3} + \frac{u^5}{5\cdot 2} - \frac{u^7}{7\cdot 6} + \cdots\right]_0^{\sqrt{t}}$$

$$= \frac{2}{\sqrt{\pi}} \left[\sqrt{t} - \frac{(\sqrt{t})^3}{3} + \frac{(\sqrt{t})^5}{5\cdot 2} - \frac{(\sqrt{t})^7}{7\cdot 6} + \cdots\right]$$

$$= \frac{2}{\sqrt{\pi}} \left[t^{\frac{1}{2}} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{10} - \frac{t^{7/2}}{42} + \cdots\right]$$

$$\therefore \operatorname{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \left[t^{\frac{1}{2}} - \frac{t^{\frac{3}{2}}}{3} + \frac{t^{5/2}}{10} - \frac{t^{7/2}}{42} + \cdots\right]$$

Taking L.T on both sides, we get

$$L\{\operatorname{erf}(\sqrt{t})\} = L\left\{\frac{2}{\sqrt{\pi}} \left[t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{10} - \frac{t^{7/2}}{42} + \cdots\right]\right\}$$

$$= \frac{2}{\sqrt{\pi}} \left\{L(t^{1/2}) - \frac{1}{3} L(t^{3/2}) + \frac{1}{10} L(t^{5/2}) - \cdots\right\}$$

$$= \frac{2}{\sqrt{\pi}} \left\{\frac{\Gamma(\frac{1}{2}+1)}{p^{\frac{1}{2}+1}} - \frac{1}{3} \frac{\Gamma(\frac{3}{2}+1)}{p^{3/2+1}} + \frac{1}{10} \frac{\Gamma(\frac{5}{2}+1)}{p^{5/2+1}} - \cdots\right\}$$

$$= \frac{2}{\sqrt{\pi}} \left\{\frac{\frac{1}{2}\sqrt{\pi}}{p^{3/2}} - \frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{3p^{5/2}} + \frac{1}{10} \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{p^{7/2}} - \cdots\right\}$$

$$= \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2}\sqrt{\pi} \cdot \frac{1}{p^{3/2}} \left\{1 - \frac{\frac{3}{2}}{3p} + \frac{\frac{5}{2}\cdot\frac{3}{2}}{10p^2} - \cdots\right\}$$

$$= \frac{1}{p^{3/2}} \left\{1 - \frac{1}{2}\left(\frac{1}{p}\right) + \frac{1}{2}\cdot\frac{3}{4}\left(\frac{1}{p}\right)^2 - \cdots\right\}$$

$$= \frac{1}{p^{3/2}} \left\{1 + \frac{1}{p}\right\}^{-\frac{1}{2}}$$

$$= \frac{1}{p^{3/2}} \left\{\frac{p+1}{p}\right\}^{-\frac{1}{2}} = \frac{1}{p^{3/2}} \left\{\frac{p}{p+1}\right\}^{\frac{1}{2}}$$

$$= \frac{1}{p\sqrt{p}} \cdot \frac{\sqrt{p}}{\sqrt{p+1}} = \frac{1}{p\sqrt{p+1}}$$

$$\therefore L\{erf(\sqrt{t})\} = \frac{1}{p\sqrt{p+1}} = f(p)$$

By change of scale property

If $L\{F(t)\} = f(p)$, then $L\{F(at)\} = \frac{1}{a} f\left(\frac{p}{a}\right)$

$$\therefore L\{erf(2\sqrt{t})\} = L\{erf\sqrt{4t}\} \quad ; a = 4$$

$$= \frac{1}{4} f\left(\frac{p}{4}\right)$$

$$= \frac{1}{4} \frac{1}{\frac{p}{4}\sqrt{\frac{p}{4}+1}}$$

$$= \frac{2}{p\sqrt{p+4}}$$

$$\therefore L\{erf\, 2\sqrt{t}\} = \frac{2}{p\sqrt{p+4}}$$

we know that

$$L\{t\,F(t)\} = -\frac{d}{dp} L\{F(t)\}$$

$$\Rightarrow L\{t\ erf(2\sqrt{t})\} = -\frac{d}{dp} L\{erf\, 2\sqrt{t}\} = -\frac{d}{dp}\left[\frac{2}{p\sqrt{p+4}}\right]$$

$$= -\left[\frac{p\sqrt{p+4}\,(0) - 2\frac{d}{dp}p\sqrt{p+4}}{p^2(p+4)}\right]$$

$$= \frac{2\left\{\sqrt{p+4} + p\frac{1}{2\sqrt{p+4}}\right\}}{p^2(p+4)}$$

$$= \frac{2\sqrt{p+4}}{p^2(p+4)} + \frac{p}{p^2(p+4)\sqrt{p+4}} = \frac{2(p+4)+p}{p^2(p+4)\sqrt{p+4}}$$

$$= \frac{3p+8}{p^2(p^2+4)^{3/2}}$$

$$\therefore \boxed{L\{t\ erf(2\sqrt{t})\} = \frac{3p+8}{p^2(p^2+4)^{3/2}}}$$

Laguerre polynomial: The Laguerre polynomial is denoted by $L_n(t)$ and defined as $L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}\left[e^{-t}t^n\right]$, $n = 0, 1, 2, 3, \ldots$

P. Find L.T of $L_n(t)$

Sol: we know that

$$L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}\left[e^{-t}t^n\right];\ n = 0, 1, 2, \ldots$$

Now

$$L\{L_n(t)\} = L\left\{\frac{e^t}{n!}\frac{d^n}{dt^n}\left[e^{-t}t^n\right]\right\}$$

$$= \int_0^\infty e^{-pt}\,\frac{e^t}{n!}\frac{d^n}{dt^n}\left[e^{-t}t^n\right]dt$$

$$= \frac{1}{n!}\int_0^\infty \underbrace{\left[e^{-(p-1)t}\right]}_{f}\underbrace{\left[\frac{d^n}{dt^n}\left[e^{-t}t^n\right]\right]}_{g}dt$$

$$= \frac{1}{n!}\left\{\left[e^{-(p-1)t}\frac{d^{n-1}}{dt^{n-1}}\left(e^{-t}t^n\right)\right]_0^\infty + \int_0^\infty (p-1)\,e^{-(p-1)t}\frac{d^{n-1}}{dt^{n-1}}\left[e^{-t}t^n\right]dt\right.$$

$$= \frac{1}{n!}\left\{(p-1)\int_0^\infty e^{-(p-1)t}\frac{d^{n-1}}{dt^{n-1}}\left[e^{-t}t^n\right]dt\right\}$$

By proceeding similarly again and aagin, we get

$$L\{L_n(t)\} = \frac{(p-1)^n}{n!}\int_0^\infty e^{-(p-1)t}e^{-t}t^n\,dt$$

$$= \frac{(p-1)^n}{n!}\int_0^\infty e^{-pt}e^{t}e^{-t}t^n\,dt$$

$$= \frac{(p-1)^n}{n!}\int_0^\infty e^{-pt}t^n\,dt$$

$$= \frac{(p-1)^n}{n!}L\{t^n\}$$

$$= \frac{(p-1)^n}{n!}\frac{\Gamma(n+1)}{p^{n+1}} = \frac{(p-1)^n}{n!}\cdot\frac{n!}{p^{n+1}}$$

$$\therefore L\{L_n(t)\} = \frac{(p-1)^n}{p^{n+1}}$$

The Exponential Integral: The exponential integral is defined as $E(t) = \int_t^{\infty} \frac{e^{-x}}{x}\, dx$

P. Find Laplace Transform of $E(t)$.

Sol: we know that: $E(t) = \int_t^{\infty} \frac{e^{-x}}{x}\, dx = F(t)$ say

Then $\lim_{t \to \infty} F(t) = 0$ and

$$F(t) = -\int_{\infty}^{t} \frac{e^{-x}}{x}\, dx$$

$$\Rightarrow F'(t) = -\frac{e^{-t}}{t}$$

$$\Rightarrow t F'(t) = -e^{-t}$$

$$\Rightarrow L\{t F'(t)\} = -L\{e^{-t}\}$$

$$\Rightarrow -\frac{d}{dp} L\{F'(t)\} = -\frac{1}{p+1}$$

$$\Rightarrow -\frac{d}{dp}\{p f(p) - F(0)\} = -\frac{1}{p+1}$$

$$\Rightarrow \frac{d}{dp}\, p f(p) = \frac{1}{p+1}$$

on integrating, we get

$$\int \frac{d}{dp}\, p f(p)\, dp = \int \frac{1}{p+1}\, dp$$

$$\Rightarrow p f(p) = \log(p+1) + c \longrightarrow (1)$$

Taking limit $p \to 0$, we get

$$\Rightarrow \lim_{p \to 0} p f(p) = \lim_{p \to 0} [\log(p+1) + c]$$

$$\Rightarrow 0 = 0 + c \Rightarrow c = 0$$

Then eq (1) $\Rightarrow$ $p f(p) = \log(p+1)$

$$\Rightarrow f(p) = \frac{\log(p+1)}{p}$$

$$\Rightarrow L\{F(t)\} = \frac{\log(p+1)}{p}$$

$$\Rightarrow L\{E(t)\} = \frac{\log(p+1)}{p}$$

P. Show that i) $L\{\sinh at \cdot \cos at\} = \dfrac{a(p^2-2a^2)}{p^4+4a^4}$

ii) $L\{\sinh at \cdot \sin at\} = \dfrac{2a^2p}{p^4+4a^4}$

Sol: Take $F(t) = \sinh at \Rightarrow L\{F(t)\} = L\{\sinh at\}$

$$= \frac{a}{p^2-a^2} = f(p)$$

By First Shifting Theorem

if $L\{F(t)\} = f(p)$, then $L\{e^{at}F(t)\} = f(p-a)$

Then $L\{e^{iat}F(t)\} = f(p-ia)$

$$= \frac{a}{(p-ia)^2-a^2}$$

$$= \frac{a}{p^2-a^2-2ipa-a^2} = \frac{a}{(p^2-2a^2)-i(2pa)}$$

$$= \frac{a}{(p^2-2a^2)-i(2pa)} \times \frac{(p^2-2a^2)+i(2pa)}{(p^2-2a^2)+i(2pa)}$$

$$= \frac{a[(p^2-2a^2)+i(2pa)]}{(p^2-2a^2)^2+(2pa)^2}$$

$$= \frac{a[(p^2-2a^2)+i(2pa)]}{p^4+4a^4-4p^2a^2+4p^2a^2}$$

$$= \frac{a[(p^2-2a^2)+i(2pa)]}{p^4+4a^4} = \frac{a(p^2-2a^2)}{p^4+4a^4} + i\frac{2pa^2}{p^4+4a^4}$$

$$\therefore L\{e^{iat}F(t)\} = \frac{a(p^2-2a^2)}{p^4+4a^4} + i\frac{2pa^2}{p^4+4a^4}$$

i.e $L\{[\cos at + i\sin at]\sinh at\} = \dfrac{a(p^2-2a^2)}{p^4+4a^4} + i\dfrac{2pa^2}{p^4+4a^4}$

Equating Real and Imaginary parts on both sides

$$L\{\sinh at \cos at\} = \frac{a(p^2-2a^2)}{p^4+4a^4}$$

$$L\{\sinh at \sin at\} = \frac{2pa^2}{p^4+4a^4}$$

P. Find $L\{(1+te^{-t})^3\}$

Now

$$L\{(1+te^{-t})^3\} = L\{1+3te^{-t}+3t^2e^{-2t}+t^3e^{-3t}\}$$

$$= L(1)+3L(te^{-t})+3L(t^2e^{-2t})+L(t^3e^{-3t})$$

$$= \frac{1}{p}+3(-1)\frac{d}{dp}L(e^{-t})+3(-1)^2\frac{d^2}{dp^2}L(e^{-2t})+ (-1)^3\frac{d^3}{dp^3}L(e^{-3t})$$

$$= \frac{1}{p}-3\frac{d}{dp}\left(\frac{1}{p+1}\right)+3\frac{d^2}{dp^2}\left(\frac{1}{p+2}\right)-\frac{d^3}{dp^3}\left(\frac{1}{p+3}\right)$$

$$= \frac{1}{p}+\frac{3}{(p+1)^2}+\frac{6}{(p+2)^3}+\frac{6}{(p+3)^4}$$

$$\therefore L\{(1+te^{-t})^3\} = \frac{1}{p}+\frac{3}{(p+1)^2}+\frac{6}{(p+2)^3}+\frac{6}{(p+3)^4}$$

= Ans.